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EFFECTIVE TEACHING AND LEARNING OF MATHEMATICS THROUGH BRIDGING THEORY AND PRACTICE

Editors:

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EFFECTIVE TEACHING AND LEARNING OF MATHEMATICS THROUGH BRIDGING THEORY AND PRACTICE

UČINKOVITO POUČAVANJE I UČENJE MATEMATIKE KROZ POVEZIVANJE TEORIJE I PRAKSE

monograph

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Osijek, 2024

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A word from the Editorial Board

Teaching and learning mathematics dates back to Antics. Mesopotamian and Egyptian scribes mastered arithmetic techniques as a requirement for their future practice, and in ancient Greece, general education included rhetoric and mathematics. Before the twentieth century, only the privileged class studied mathematics, and the general population learned only basic arithmetic. Though access to mathematics education was limited, teaching approaches varied even centuries ago (Ackerberg-Hastings, 2014). In Renaissance schools, multiple students studied different topics simultaneously. Notes from the eighteenth century show that students recognised and valued interesting lecturers who added theatrics to their lessons rather than reading from their scripts. Child-centred teaching and inquiry-based learning ideas date back to the nineteenth century. Throughout history, new technology has altered classroom practice. Blackboards and notebooks allowed opportunities to engage students in written work more than oral; textbooks and manipulatives enabled individual and hands-on work; new media, television and the internet reduced teachers' work and even raised the possibility of replacing them. The initial inquiry into effective learning in the twentieth century noted the poor outcomes of repetition and rote learning, inspiring the New Math programme to formalise mathematics education. Despite the final rejection of the formalised aspect of teaching, the movement severely affected the development of a structured mathematics curriculum and the accessibility of mathematics education (Kilpatrick, 2014).

Effective teaching and learning differ across ages and cultures. Modern society requires educating citizens for the contemporary work market, developing students' problem-solving skills, creative and divergent thinking, teamwork, and social competencies. The notions of mathematical literacy and mathematical proficiency characterise the result of effective learning from different perspectives (Niss & Jablonka, 2020). The former relates to the role of mathematics in everyday situations, particularly applying mathematics in advancing citizenship, sustainability and prosperity. The latter relates to mastering mathematical knowledge and skills, including mathematical understanding and reasoning, communication and representation, problems posing and solving, mathematical modelling, and others. Learning mathematics exceeded correctly solving a range of tasks; students should appreciate mathematics as interesting, useful and valuable. Bryan et al. (2007) examined the views of teachers with different socio-cultural backgrounds on effective teaching and learning. Quality teaching depends on instruction grounded in solid mathematical knowledge, the ability to purposefully use a variety of methods and tools depending on students' needs, and to build students' interest. Preferred teaching methods vary regarding cultural background, cautioning that factors outside classroom condition practices within the classroom. The notion of effective teaching and learning is thus multifaceted. According to Anthony and Walshaw (2009), effective mathematics teaching is responsive to students' different needs and aims to develop a range of mathematical competencies, social skills and positive attitudes toward mathematics.

The psychology of learning had an impact on early research in mathematics education, hence they were focused on understanding how an individual constructs knowledge. Experimental studies explored how variations in teaching affect learner's performance to constitute what is effective teaching. Theory and research development acknowledged the multiplicity of factors influencing teaching and learning mathematics, such as social and cultural environment, curricular requirements, use of artefacts, representations and technology, students' and teachers' attitudes and beliefs, and others. Though the scope of mathematics education research broadened to encompass the themes at stake, exploring and developing mathematics education practice and communicating the scientific knowledge to practitioners remain relevant issues (Bakker et al., 2021).

Mathematics education research can inform the practice on different levels and with different effects. International studies, such as TIMSS and PISA, have a significant influence on educational policies. Experimental studies inform about the benefits of particular instructions, for example, directed instruction proved helpful for improving deep procedural knowledge (Inglis & Foster, 2018). Studies showed that effective implementations of manipulatives, textbooks and technology in classroom practice depend on the teacher's knowledge and skills in careful instructional design. Researcher distances themselves from the teaching practice; using theoretical and methodological lenses enables them to observe and evaluate issues at stake objectively and comprehensively (Chevallard & Bosch, 2014), whereas practitioners reflect on particular teaching or learning moment (Huillet, 2014).

Bridging theory and practice concerns the responsibility for communication and collaboration between educational researchers, stakeholders and practitioners to influence and improve mathematics education. Teacher education and professional development are a direct line of disseminating research, but without longitudinal support and studies, there is no evidence of their impact (Bakker et al., 2021). Examining and designing research-based textbooks and educational tools is another mode of implementing theory into practice. Project-based research with teachers as recipients, participants or collaborators proved useful for developing researchers' and teachers' professional competence (Kieran et al., 2013). The disruption caused by the 2020 pandemic corroborated the precedence of direct over online teaching. Both sides of the bridge benefit from linking theory and practice, but for the research to effectively impact teaching and learning, connections need to be pertinent and transparent to both included parties.

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EFFECTIVE TEACHING AND LEARNING OF MATHEMATICS THROUGH BRIDGING THEORY AND PRACTICE

monograph

Preface

Research developments, digital technology incorporation, and inventive teaching techniques are the main forces behind the ongoing change in the mathematics education scene. Bridging the gap between theory and practice in the classroom is the main thrust of this monograph, *Effective Teaching and Learning of Mathematics through Bridging Theory and Practice*. It compiles a variety of papers that highlight innovative methods and practical ideas for teaching mathematics, spanning from early childhood to university level.

The importance of creative teaching methods throughout the early years of mathematical learning is emphasised in the first chapter, *Advancing Mathematics Education: New Strategies and Collaborative Efforts*. The first paper shows how interactive techniques can improve young learners' knowledge by discussing the function of dialogic teaching in combinatorial circumstances within preschool settings. The preschool years are vital for introducing mathematical ideas because they allow kids to begin with concrete, material-based learning that moves towards abstraction, and combinatorial notions support the development of generalisation and systematic thinking abilities. In order to meet the various educational needs of low-achieving students, the authors of the second paper in this chapter emphasise the significance of cooperation between mathematics teachers and special educators. In the third paper, the authors analyse superficial strategies for solving compare-combine problems and provide strategies to overcome common misconceptions. The authors of the final paper examine problem-posing exercises for second-graders to gain insights towards developing critical thinking and problem-solving abilities.

The second chapter, *Development of Mathematical Thinking through Theory, Modelling, and Activity*, focuses attention on how mathematical thinking develops throughout the course of different educational levels. The authors of the first paper investigate how exercises in number theory improve mathematical cognition. The second paper illustrates the possibility of implementing problem-solving activities by presenting Fermi problems as a tool for mathematical modelling in secondary school. The third paper's authors emphasise the need for visual aids in early mathematics education, using mathematical picture books to help first-graders understand the concept of zero. In the fourth paper, the authors provide a critical analysis of the present teaching resources while looking at word problems in Croatian elementary mathematics textbooks. The last paper in this chapter explores the use of performing arts in preschool mathematics education, showing innovative approaches to make learning more interesting and successful.

The third chapter, *Exploring Geometric Concepts and Spatial Abilities in Mathematics*, relates to the visual and spatial dimensions involved in learning. The first paper focuses on the overall classroom climate of geometry courses. The second paper reports how a supportive learning environment is shaped by using statistics combined with spatial abilities to generate multimodal mathematics education. The third paper's author examines student-generated drawings in math and provides insights into the creative processes that make for better understanding. As part of the fourth paper, the author uses geometrical algebra and the three reflection theorem in the research of Kárteszi points of a triangle, where an extensive, detailed theoretical review is made of geometrical ideas.

The fourth chapter, *Contemporary Challenges and Solutions in Mathematics Education*, examines the integration of teaching mathematics through different paradigms and approaches. In the first paper, the authors investigate how learning paradigms configure teaching processes in the didactic traditional-contemporary construct. The second paper is dedicated to the application of decision trees in research in primary school mathematics and offers new analytical solutions. The authors of the third paper review the ideas of hybrid teaching approaches and further check their influences on the outcome of state graduation exams. In the invitation to advance cognitive development in higher education, the fourth paper presents an analysis of the logical operational and reasoning capacities of university students. Concepts that require new initiatives for lifetime learning in dealing with mathematics for mathematics teachers are considered in the last paper of the chapter, "Enactive Learning in Mathematics".

The last chapter, *Digital Learning Environments and Their Impact on Mathematics Education*, addresses technological use in the study of mathematics. The authors of the first paper discuss digital-game-based learning and innovative methods in mathematics, making the subject enjoyable and interactive. On the other hand, the authors of the second paper call for practical skills demonstrated by a study on advanced educational methods and the application of mathematical teachers' knowledge during classes with advanced programmers. Synthesising, the authors of the third paper view general trends, perspectives, challenges, and problems of digital pedagogy as part of a broad perspective on the current digital landscape in mathematics education. The last paper in this chapter concentrates on sharing the deep details of implementing a Tablet-Human Hybrid Model of Avatar within the context of face-to-face university mathematics courses to provide some speculation on the future of hybrid education.

This monograph represents a collective effort of researchers from various backgrounds, united by the common goal of advancing mathematics education. We hope that the insights and findings presented in this collection will inspire educators, researchers, and policymakers to explore new horizons and continue innovating in the field of mathematics education.

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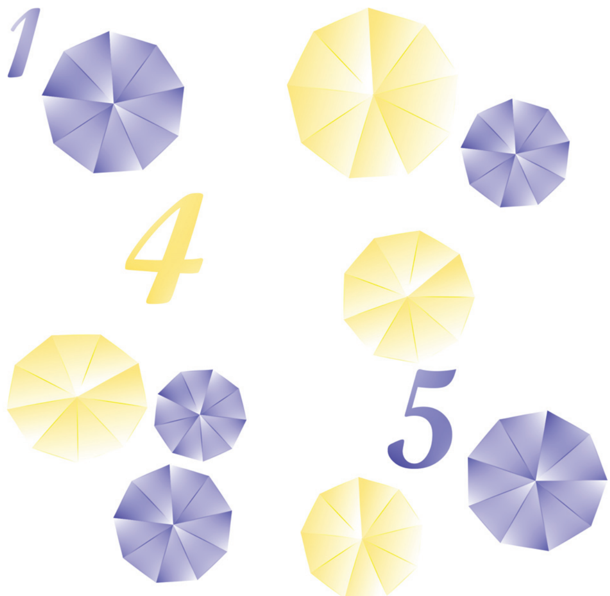
1.
Advancing Mathematics Education: New Strategies and Collaborative Efforts



Preschool Teacher's Planning and Implementation of Dialogic Teaching in Dealing with Combinatorics Situations

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Abstract. The preschool years, especially if a child attends kindergarten, are the entry point to the systematic acquisition of mathematical concepts. In this period, these are linked to material reality, but in further learning the concepts become increasingly detached from it, the concepts becoming abstract mental objects. In addition to some of the most commonly covered topics in the preschool period (e.g. counting, geometric shapes), research on early mathematics learning emphasises the importance of learning about simple combinatorial situations. Combinatorics concepts, when represented and conceptualised appropriately, support the child's development of generalisation and systematic thinking skills, while at the same time being quite independent content, i.e. not directly linked to the child's knowledge of other mathematical content. Although the teacher and the preschool teacher in the "modern" (as opposed to the "traditional") conception of teaching and learning are systematically being removed from their teaching role, and there is still a concern in the field of preschool education to "teach the children a lesson", we have decided to highlight or refresh the importance of the preschool teacher's role in the kindergarten. In this paper, we argue that the preschool teacher, as a knowledge holder, has a key role to play in the teaching of mathematics in order for children to progress in their knowledge. One of the hallmarks of good teaching is the thoughtful planning and implementation of dialogue with children in the context of dialogic teaching, which practically never stands alone, but is intertwined with direct teaching – the teacher also directly transmits certain knowledge, terms, procedures, etc., as part of his/her work. In this paper, we are interested in the quality of the teacher's implementation of combinatorics activities in terms of some factors (the type and appropriateness of the representations used, the teacher's management of the task, the teacher's planning and implementation of the dialogue with the children) that influence the success of the implementation of combinatorics activities in the preschool period. The sample included a group of 11 preschool teachers who had acquired relevant theoretical knowledge on conducting dialogue with preschool children in the context of a course on early learning of mathematics, on the basis of which they planned and implemented activities in kindergarten. Insights into the quality of activity implementation were gained by coding and qualitatively analysing transcribed recordings of the teacher's dialogue management with children in the process of implementing mathematics activities, and by comparing the planning of the dialogue management and its implementation. The contribution of our paper is that, unlike research that mainly examines children's reasoning in solving selected combinatorics problems, our study highlights the role of the teacher as an essential actor in the process of children's mathematics learning. Furthermore, the paper will critically highlight the understanding of quality activity management with preschool children in the light of teaching and learning theory.

Keywords: combinatorics, dialogic teaching, mathematics, preschool child, preschool teacher

1. Introduction

Just like a teacher, a preschool teacher is a bearer of knowledge, which he or she must impart to children in an appropriate way. Imparting is understood here in the sense that the knowledge is beyond the individual and can be acquired by the child through the appropriate approach of the preschool teacher. An approach to teaching is an imaginative combination of methods and forms of working to address a particular topic or concepts, for a particular group of learners or children. We would like to point out from the outset that quality teaching is not that which is labelled "modern", but must be based on consideration of the child, the content, the objectives, the methods and forms of work and other factors that are relevant in a given learning situation. The label "modern" teaching aims above all to make a distinction with "traditional" teaching, which is less desirable, leading to an understanding of the role of the teacher as a co-learner or facilitator of learning. Biesta (2017) contradicts this when he writes: "...the teacher should be understood as someone who, in the most general sense, brings something new to the educational situation, something that has not been present before" (p. 74).

> "One thing that teachers and those concerned with teaching can do is to resist and break with the constructivist "common sense" about teaching – namely the "common sense" in which the teacher is the one who has nothing to give, who is there to draw out of the pupil what is already there, who is there to facilitate the pupils' learning, not to teach them anything, who is there to make the learning process as smooth and enjoyable as possible, who is there not to ask difficult questions or introduce difficult concepts in the hope that the pupils will go away as satisfied customers." (Biesta, 2017, p. 93)

We cannot parallel kindergarten with school, but we can say that kindergarten is also a place where children acquire knowledge, where they learn. The differences

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between preschool and school are clear from an educational point of view, and we do not dispute them, but at the same time we do not agree that free play is the most important aspect of preschool, and that the role of the preschool teacher has to be removed from the role of teaching. Preschool teacher has highly interactive role in the process of child's learning. The child's development will be static unless he/she is able to work in his/her zone of proximal development[1]. In Vygotsky's framework teaching does not wait upon development but propels it.

> "Teachers need to know the learners well, so they can provide the right level of guidance, and gradually withdraw it as the child comes to understand and perform the task alone. Teachers cannot leave children to discover the world alone in free play." (Pardjono, 2002, p. 175)

The preschool teacher's teaching role will be understood in the sense that the preschool teacher brings something new to the group of children, something that the children do not know yet but could, depending on their abilities and interests. The concern that this is a case of "schooling" the children, which is often expressed in Slovenia (and nowhere explicitly stated), is unfounded if the preschool teacher brings new knowledge to the group in an appropriate way. When we say "brings" we do not mean the passive role of the children (unfortunately, space does not allow us to focus on the passive-active dichotomy, which is a new construct, not as simple as it looks at first sight). Starting from the theoretical assumptions of Vygotskian theory, teamwork between the preschool teacher and the children in the preschool period is very important because children will not be able to advance in their zone of proximal development unless they have an opportunity to share in joint interactions with a teacher who has sensitivity to children's changing knowledge (source). "Therefore, in the Vygotskian classroom, teacher doesn't simply wander around the classroom scanning children's activities and making occasional comments or directing questions to the child." (Pardjono, p. 175)

2. Dialogic teaching

Hattie et al. (2017) distinguish between direct and dialogic teaching, going beyond the dichotomy of desirable, undesirable. The purpose of the distinction between the two teaching approaches is to distinguish the essential characteristics of each, which are complementary, or one is more appropriate and the other less appropriate in a given learning situation. We will not make the distinction here, but will concentrate on dialogic teaching. In dialogic teaching, the role of *talk* is fundamental to both knowing and learning mathematics; *group work* provides a venue for more talking and listening; *the sequencing of topics* dictated by both disciplinary and developmental progressions; *instructional tasks* initiate to new ideas and deepen children's understanding of concepts and help them become more competent with

[1] The zone of proximal development is defined as the distance between the actual developmental level as determined by independent problem solving and level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers (Vygostski, 1978).

what they already know; *the role of the feedback* in not merely correcting misconceptions, but advancing children's growing intellectual authority about how to judge the correctness of one's own and others' reasoning; *creativity* – children should make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove these conjectures; *the importance of diagnosing children's thinking* for building on children's initial thinking to move forward important ideas of the discipline; children's participation in the *introduction and role of definitions; representations* are used not only for illustrating mathematical ideas, but also for thinking with using them (Hattie et al., 2017).

In Slovenia, a fairly established teaching method is conversation, which, from the point of view of ordinary discourse, does not necessarily represent a dialogue process in teaching. As Alexander (2008) argues, the essential aspect that distinguishes conversation from dialogue is that classroom dialogue explicitly seeks to make attention and engagement mandatory and to chain exchanges into a meaningful sequence. The act of questioning is a key point that differentiates conversation from dialogue, and the critical issue is what follows from answers. If the answer gives rise to a new question from itself, then we speak about dialogue (Bakhtin, 1986). For a better understanding of what dialogue is, following Alexander (2008) we will summarise what dialogue is not (p. 105):

1. interactions tend to be brief rather than sustained, teachers move from one child to another in rapid succession in order to maximise participation, rather than developing sustained and incremental lines of thinking and understanding;
2. teachers ask closes questions;
3. children concentrate on identifying "correct" answers, and teacher gloss over "wrong" answers rather than use them as stepping stones to understanding;
4. the questions are "test" rather than "authentic";
5. feedback tend to encourage and praise rather than inform, the cognitive potential of exchange is lost.

Dialogue is not tied to a specific form of work. It can be present in whole class teaching, guided group work, collaborative group work, individual work, pair work. The teacher's organisation of interactions, how the children themselves talk, and the forms of oral expression and interaction that they need to experience and eventually master are essential. The learning talk repertoire includes the ability to narrate, explain, instruct, ask different kind of questions, receive, act and build upon answers, analyse and solve problems, speculate and imagine, explore and evaluate ideas, discuss, argue, reason and justify and negotiate together with four contingent abilities that are vital if children are to gain the full potential of talking to others: to listen, to be receptive to alternative viewpoints, to think about what they hear and to give others to think (Alexander, 2008). Children can learn the skills of speaking, of engaging in dialogue both with the teacher and with other children, if the teacher is aware of the factors of successful interaction and integrates them thoughtfully and meaningfully into his/her work. Asking questions or conducting a dialogue is by no means a matter of inspiration that will spontaneously arise on the occasion of working with children, but requires reflection on the questions, the way they are asked, the role of the children, the chaining of questions... This is particularly true for mathematical topics, where, in addition to didactical knowledge, expertise is also needed to ask questions appropriately. This is not, of course, about introducing complex mathematical discourse into the preschool period, but it is important that the preschool teacher expresses himself appropriately and consistently when conducting a dialogue with the children. In addition to the appropriate use of terms (e.g. we do not expect an preschool teacher to use the term permutation in combinatorics content if he/she is discussing with children the arrangement of 3 different coloured wagons into different compositions, but to address the concept appropriately and use it consistently), it is necessary to follow the criteria of a quality dialogue that provides the best chances for children to develop the diverse learning talk repertoire. These are (Alexander, 2008): dialogue should be *collaborative* – teachers and children address learning, tasks together, whether as a group or as a class, *reciprocal* – teachers and children listen to each other, share ideas and consider alternative viewpoints, *supportive* – children articulate their ideas freely, without a fear of embarrassment over "wrong" answers; and they help each other to reach common understandings, *cumulative* – teachers and children build on their own and each other's ideas and chain them into coherent lines of thinking and enquiry, and *purposeful* – teachers plan and guide classroom talk with specific educational goals in view.

3. Teaching and learning combinatorics in preschool

Combinatorics is the branch of mathematics that deals with the counting and ordering of the elements of a given finite set. By solving problems in combinatorics, pupils/children develop the concept of counting, make connections between concepts, learn generalisation in mathematics, optimisation and systematic thinking (Bräuning, 2019; English 1991, 2005). Pessoa and Borba (2009, in Borba et al., 2021) argue that knowledge in combinatorics is developed from the early years of schooling and that children should be provided with activities that involve all the different combinatorial situations possible: the combinatorial rule of product, permutations, variations and combinations. They justify this on the grounds that these four types of combinatorial situations contain the basic ideas of combinatorics, and if children are brought into contact with basic situations that represent these contents, the child is enabled to develop combinatorial thinking.

Several authors (Bräuning, 2019; Lockwood et al., 2020; Zapata-Cardona, 2018) stress the importance of combinatorics content and its inclusion in the mathematics curriculum. Research results (English 1991, 2005; Palmér and van Bommel, 2016; van Bommel and Palmér, 2018) have shown that combinatorics content is suitable to be introduced already in the preschool period. Several arguments have been given in the literature for this: combinatorics content is independent of some other mathematics content (e.g., computation), often requiring only the counting of all possibilities (English, 2005; Lockwood et al., 2020), and it does not involve complex mathematical terms. Problems in combinatorics can be solved in different ways – using different representations (Bräuning, 2019; English, 2005; Lockwood, 2020), including concrete representations, to adjust their complexity (English, 2005). It is also relatively easy to relate the content of combinatorics to problem solving in children's everyday play and to situations in everyday life (English, 2005; Krekić-Pinter et al., 2015; Lockwood, 2020).

Preschool teachers need to have knowledge of combinatorics in order to be able to carry out quality and successful combinatorics activities. Several research results support the view that children in preschool are already capable of solving combinatorial problems if they are properly planned and guided by the preschool teacher. Borba et al. (2021), similarly to English (2005) and Zapata-Cardona (2018), found that preschool children are already capable of systematically solving combinatorial problems if they are presented in a realistic, meaningful context and supported by an appropriate explanation that aims at teaching them to systematically search for possibilities, to clearly define what the possibilities are in a given situation (which elements to choose from), to distinguish between different possibilities, how to differentiate between them, how to recognise their equality, and whether the order in which the elements are chosen is important. Preschool teachers also need to pay attention to the child's distinction between counting the elements of a set and counting the possibilities that he or she forms from the given elements of the set (Zapata-Cardona, 2018). Bräuning (2019) says that it is also necessary to take into account that the child's way of expressing mathematical thinking is different and limited in terms of verbal abilities (Bräuning, 2019). It is important that the preschool teacher familiarises the child with the task and the language used in guiding the problem solving (English, 1991). While guiding the child to solve the task, he/she should also ask relevant questions to help the child to make progress in solving the task and to solve it successfully (English, 2005; Zapata-Cardona, 2018). In her research, Zapata-Cardona (2018) found that without adequate support from the preschool teacher with questions that guide the child to find the final solution, children are not as successful and make more mistakes. Bräuning (2019) reports similar results in her research, finding that by repeatedly solving the task with appropriate support from the preschool teacher, children developed a more systematic way of finding all possible arrangements of the three elements of a set by asking questions over time. Some children were also able to justify why they had found all possible arrangements.

Another important aspect that affects the quality of the activity is the representations used to represent and solve the combinatorics problem. According to Zapata-Cardona (2018), it is very important for children to get adequate support in the use of concrete representations to illustrate the different possibilities when solving the problem. Concrete representations help children to communicate mathematical ideas, stimulate their thinking and help them to find a solution to a problem without being told how to solve it. Borba et al. (2021) found in their study that children who solved combinatorics problems made more progress when they were offered a semi-concrete representation in the form of illustrative cards (manipulative material) than those who had to create the representation themselves (semi-concrete or semi-abstract). Fessakis and Kafoussi (2009) also carried out a study on the role of representations in solving mathematical problems in combinatorics. The study included 30 preschool children. They were interested in how successful they were in solving combinatorics problems using a computer application (solving in an ICT environment) and how well they did using concrete tools. The results of the study showed that there were no statistically significant differences in performance between the two groups of children who solved the combinatorial problem in two different ICT environments in terms of the level of support. However, it turned out that the two groups of children working in two different ICT environments then performed statistically significantly better on the combinatorial problem with concrete tools than in the ICT environment. In this respect, Fessakis and Kafoussi (2009) suggest two possible reasons for the higher performance on the combinatorial problem with concrete tools. The first is that the children have had prior experience with combinatorial situations, having solved the problem in an ICT environment. The second is that children at preschool level are more successful in solving combinatorial problems if they manipulate concrete objects.

Based on the results of the above research, we argue that in order to plan and implement quality teaching of combinatorics content, the preschool teacher needs to have both mathematical content, pedagogical and didactical knowledge. He/she must be familiar with different types of combinatorial situations and understand how to solve them not only at the symbolic level, but also at the concrete and graphic level. From the point of view of pedagogical and didactical knowledge, it is particularly important to be able to select appropriate combinatorial problems, to support them with an appropriate representation and to guide the solving of the problem. In order to successfully guide children through the problem-solving process, the preschool teacher also needs to know what terminology to use, what questions to ask the children and how to engage in a good quality dialogue with them.

4. Defining the research problem

Lockwood et al. (2020) note that there is little research on combinatorics content that addresses how preschool teachers deliver instruction and how they support children in solving combinatorial problems. Existing research in the area of teaching combinatorics content in preschool focuses on identifying the strategies children use to solve combinatorial problems and the difficulties they face in doing so (Borba et al., 2021; Bräuning, 2019; English, 1991, 2005; Zapata-Cardona, 2018), on the type of representations they use when solving combinatorial problems (van Bommel and Palmér, 2016), and on the effects of teaching combinatorial problem solving on solving performance (Borba et al., 2021; Fessakis and Kafoussi, 2009; van Bommel and Palmér, 2018, 2021). An important finding from all of this research is that children in the preschool years already have adequately developed the operational structures necessary for successful combinatorial problem solving. The role of the preschool teacher in this context is to create an appropriate learning environment, in which the child can successfully solve combinatorial problems, and to support and guide the child appropriately. In this regard, English (1991; 2005) and Zapata-Cardona (2018) emphasise that problem solving must be supported by appropriate representation and guidance from the preschool teacher.

In the research, the findings of which we present below, we set ourselves two objectives. The first objective is to analyse some of the factors that influence the success of combinatorics activities in preschool. To this end, we analysed the following factors in combinatorics activities: the type and appropriateness of the representations used, the teacher's management of the task, and the teacher's planning and implementation of the dialogue with the children in the management of the activity. In analysing the quality of the implementation of the dialogue with the children, we were interested in the consistency of the planned questions with the implemented ones, as well as in the achievement of the merits of the dialogue, which we defined following Alexander (2018). The second aim of the research is to situate the understanding of the quality of the management of the activities with preschool children in the light of teaching and learning theory, on the basis of the insights gained about the preschool teachers' implementations the content of the combinatorics in their classrooms.

4.1. Description of the sample and data collection procedure

The sample included a group of 11 preschool teachers' reports, lesson preparations, and video records. All teachers had acquired relevant theoretical knowledge in the Early Learning of Mathematics course. The teachers had to record the activity (at least 10 min long) with a group of children (aged 4–6 years) which they judged to be the most successful according to the criteria for quality dialogue with children in kindergarten.

Insights into the teacher's implementation of the mathematical dialogue were obtained by coding and qualitatively analysing the transcribed recordings of the teacher's dialogue with the children in the process of implementing the mathematical activities. The analysis of the recordings was compared with the planning of the implementation of the activities that the individual teachers had written down in their teaching preparation.

We have analysed 11 learning activities and the corresponding 11 videos of the planned activities. Each video was first transcribed. The transcripts of the videos and the corresponding learning activities were analysed in terms of the type and appropriateness of the mathematical representations, the preschool teacher's management of the problem-solving task and the planning of the dialogue during the problem-solving task. In Table 1 we present basic information about the analysed activity recordings. For some contexts, we have also recorded the extension of the mathematical context. The extensions could be related to the same content from combinatorics or the content could have changed.

Table 1. Brief description of the activities of the images analysed.

Name of activity	**Description of activities**	**Mathematical context**	**Age of children**	**Number of children**	**Form of work**
Ice-cream_1	Children demonstrate all combinations of 2 ice cream flavours out of 4.	Combinations	5–6 years	4	Guided group
Fruit	Children show all combinations of 2 fruits out of 4.	Combinations	3–5 years	8	Guided group
Ice-cream_2 *Ice-cream_3*	Children show combinations of 2 ice cream flavours out of 3 then show all possible arrangements of the 3 ice cream flavours.	Combinations Extension: permutations	4–6 years	20	Whole class
Balls_1	Children show all combinations of 2 colours of balls out of 4.	Combinations	4–5 years	8	Guided group
Dress_1	Children demonstrate hat and scarf combinations, choosing between 3 hat colours and 3 scarf colours.	The combinatorial rule of product	4–5 years	10	Guided group
Dress_2	The combinatorial rule of product	5–6 years	12; 2	Whole class Individual	
Balls_2	Children show all possible different arrangements of the 3 colours of balls in a 3×3 grid.	Permutations	5–6 years	9	Guided group
Dress_3	Children demonstrate combinations of a headdress with a top and a bottom, choosing between 2 tops and 3 bottoms.	The combinatorial rule of product	5–6 years	6	Guided group
Dress_4 *Gifts*	Children demonstrate hat and scarf combinations, choosing between 2 colours of hat and 2 colours of scarf. Extend the activity by choosing 2 presents out of 4 for each snowman.	The combinatorial rule of product Extension: combinations	5–6 years	16	Whole class
Lunch *Bouquets*	Children demonstrate combinations of a main dish with a side dish, choosing between 2 main dishes and 3 side dishes. Extend the activity by decorating the table, making bouquets. They can choose 2 flowers from a choice of 3.	The combinatorial rule of product Extension: combinations	3–5 years	5	Guided group
Dress_5 *Tower*	The children first demonstrate the clothing combinations, choosing between two colours of shirts and two colours of trousers. Extension activity – children show all the different arrangements of the 3 colours of cubes in the tower.	The combinatorial rule of product Extension: permutations	5–6 years	7	Guided group

5. Results and interpretation

The results of our research will be presented in the following sections: the type of representation and its appropriateness for the chosen combinatorial task, the preschool teacher's management of the combinatorial task and the preschool teacher's management of the dialogue with the children.

5.1. Type and appropriateness of representations used

When solving the combinatorics task, we were interested in the context the preschool teachers chose for the planned activity (mathematical or non-mathematical), the combinatorics situation (permutations, variations, combinations or the combinatorial rule of product) and how they presented the task to the children. Furthermore, we analyse what representations the preschool teachers used and their suitability for solving the given combinatorics task. We also analyse the appropriateness of the mathematical terminology used to guide the activities in relation to the representations.

We found that the majority of preschool teachers (9 out of 11) used a non-mathematical context (see Figure 1, Figure 2, Figure 3). The most frequently used context was the dressing context (see Figure 2), where children searched for possible clothing choices (the context allows to determine the choices according to the combinatorial rule of product: the number of all choices in e.g. a two-stage decision is the product of the choices at the first and second stage, if the decision at the second stage is independent of the decision at the first stage). Other contexts used were the assembly of possible combinations of lunch, fruit plate, ice-cream (see also Table 1). In these cases, when solving the tasks, the preschool teachers expected the children to find all possible combinations: two items out of three or four (see Figure 1, Figure 5). Three preschool teachers designed the task for the children as a search for permutations of three items (see Figure 3, Figure 4). For four preschool teachers, the extension was made on the change of the combinatorial content, e.g. the task of searching for combinations was extended to a task of searching for permutations.

Figure 1. Non-mathematical contex (Ice-cream_1).

Figure 2. Non-mathematical contex (Dress_1).

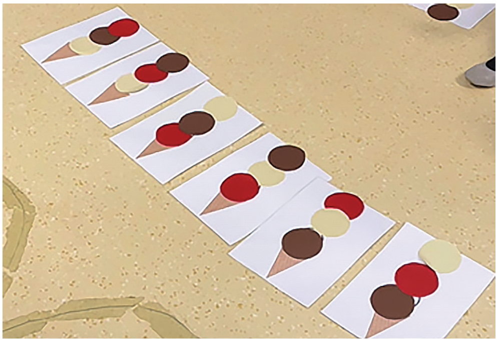

Figure 3. Non-mathematical contex (Ice-cream_3).

Figure 4. Mathematical contex (Balls_2).

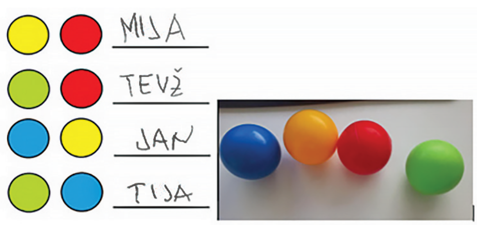

Figure 5. Mathematical contex (Balls_1).

In all the observed implementations, children solved the combinatorics task by handling concrete objects. Most of the representations were meaningfully related to the mathematical content, appropriate for the child's handling, aesthetic and allowed all the resulting possibilities to be seen simultaneously. In all the activities, there was enough material for each child to have the opportunity to handle it. In terms of the way the representations were used, in the majority of the activities (9 out of 11), the representation of all combinations or arrangements, was common. This means that each child contributed one or a few examples of combinations or arrangements to the final solution. In only two cases did each child independently create all the combinations or arrangements. In some cases, when using representations to solve the task, it was problematic that there was just as much material as there were possible combinations or arrangements. This was to prevent the child from making a mistake in figuring out how many different combinations or arrangements there were. While the initial combinatorial situation was always supported by the manipulation of concrete objects, in the extension situations it happened that these were introduced hypothetically in two cases. For example, in the *Lunch* activity, the preschool teacher asked the children how many more different combinations they would have if they added French fries as a side dish. Similarly, the extension was hypothetically made by adding the colour of the balls (activity *Balls_1)*. In both cases, the children were not able to solve the task as they did not model the possibilities with a concrete representation.

Below are some more examples of the use of representations that have shown some room for improvement.

1. Example (*Balls_1*): the preschool teacher asked the children to choose two of the four colours of balls, one being their favourite and the other their least favourite (Figure 5). At the end, they checked which children had chosen the same and which different combinations and if there were any other combinations. In this situation, it did not make sense for the child to find all the different possible combinations, as the task asked for their favourite and least favourite colour. Also, in this case the child can distinguish between "red and blue" and "blue and red" combinations, as in one case the child may like blue the most and in the other the least. The context therefore does not allow for neutrality or exclude personal preferences, which are irrelevant in combinatorics content to begin with (it could be only at the end of the activity, when e.g. all the options have been collected, that the child chooses the one he likes best). It would be better to tell the child to draw out any two colours of balls (out of a choice of four), and then the next child can draw out a combination of colours that has not yet been chosen. Here we are interested in how many children will be able to draw a different combination of ball colours.
2. Example (*Dress_3*): the preschool teacher asked the children to show in a picture how they could "dress" a girl for school (Figure 6). They had to choose between two different tops and three different bottoms. They had to take into account that the girl would be dressed differently every day. The problem situation was interesting, but each child had only one picture of the girl. This made the representation less relevant because the dressing options kept disappearing. When a child wanted to create a new option, the previous one was removed. In

order to keep track of all the options that emerged and to compare them, it is essential that the child has a view of all the options that emerged. For the given problem, there were six children in the group, so the given dressing options could be compared with each other to find all six different ones.

Figure 6. Instantaneous disappearance of the resulting combinations (Dress_3).

3. Example *(Dress_1)*: the preschool teacher asked the children to find out how many snowmen they could dress differently. They had to choose between 3 colours of hats and 3 colours of scarves. The children had twelve snowmen drawn on each sheet of paper, four snowmen on each sheet (Figure 2). Since there were four snowmen on each sheet of paper, the children did not have the possibility to handle a single snowman (the representation was in a sense fixed). The representation used thus made it significantly more difficult to compare snowmen with each other and to systematically represent the possibilities according to the fixation of the chosen elements. The preschool teacher could have given the children individual snowmen to move around during the activity in order to highlight the equality or diversity of the snowmen's dressing options and then systematically represent them in a meaningful sequence. It should also be noted that in 4 out of the 11 implementations, the representations were such as to limit the systematic display of any resulting combinations or arrangements.
4. Example (*Dress_2*): the child has filled in the given table (Figure 7) by fitting the corresponding penguin into the space with the chosen colour of scarf and hat. This represented all the possibilities, where the child could choose one scarf from three different scarf colours and combine it with one cap colour (from three). The chosen representation is a multiplicative representation of all the possibilities in the table. Although the table (Figure 7) allows for a systematic representation of all the options and their counting, the children did not design them, but only arranged them in the table. It would be better if the children first formed the options by placing the chosen combination of cap and scarf on the picture of the penguin, and then placed that option in the appropriate place in the table. The chosen way of using the table did not give the children insight into the process of creating the options they have when "dressing the penguin". However, placing the penguins in the table when they had first designed them would have allowed them to see if they had designed all the options – the table would have given them an insight into judging whether they had exhausted all the options.

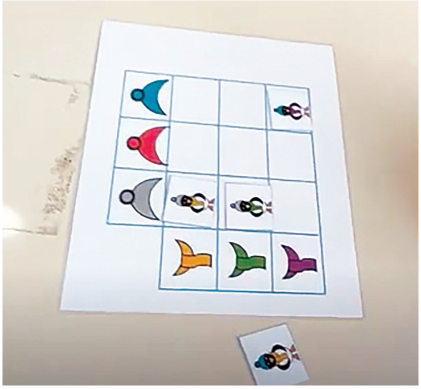

Figure 7. Fitting all possible combinations to a spreadsheet (Dress_2).

5. Example (*Balls_2*): the preschool teacher made a grid on the floor with nine (3×3) boxes for the children. The children had to arrange the three different colours of balls in the grid so that all three colours of balls were in each row and in each column. When the children had completed the grid according to this criterion, they had all possible different arrangements of the three colours in a row/column (Figure 4). It was important that the arrangements were observed from two directions, as otherwise the arrangement would be repeated. While this representation was appropriate from the point of view of use for introducing the different arrangements, it was problematic in terms of implementation. The balls used rolled off the grid and did not provide a systematic view of all the different arrangements at the end. In this activity, the children searched for the balls around the room and placed them in the grid. We also think that in this case the link to the movement activity may have distracted or discouraged the children from looking at the process of making the arrangements and the mistakes that were made. After the initial arrangement of the balls in the grid, a representation should be used that allows the individual rows or columns to be shown separately, thus providing a systematic and illustrative view of all possible arrangements of the three different colours of balls.

When presenting combinatorics activities, it is important to familiarise children with the combinatorial situation, especially with the set of elements from which they choose and create combinations or arrangements. We found that most preschool teachers (8 out of 11) took this aspect into account, asking questions at the beginning to familiarise children with the set of elements, its number and its properties. In some activities, children made combinations based on colours (e.g. *Ice-cream_1*; *Ice-cream_2*; *Balls_1*). In addition to the colours, children had to know the objects that made up the basic set. These were familiar to them as they were chosen from their everyday life (type of clothing, type of fruit and the rest of the food) and as such suitable to guide the dialogue during the activity. The children also did not appear to have any difficulties with recognising colours, so the inclusion of colour combination design was appropriate.

Another important aspect of representation is the use of mathematical terms. In combinatorics content, the most commonly used terms when asking questions are combination, arrangement, different and equal. We were interested in whether and how these terms are defined by the preschool teachers in the introductory part of the activity. We found that in 6 of the activities the term combination was not introduced. In the remaining activities, the term was introduced on a selected example, descriptively, such as *"I would like to have different combinations of snowmen, i.e. no two snowmen should be the same. No two snowmen should have the same combination of clothes"*. In only one case was an explanation with examples and counter-examples included. This means that the preschool teacher showed examples of both the same and different combinations, but did not take the opportunity to ask why the combinations were different or same. Despite the fact that in more than half of the cases the term combination was not introduced, 9 out of 11 preschool teachers used it to ask questions when leading the activity. In two of the activities, it was observed that the preschool teachers did not use the term combination, as they moved away from this content when conducting the activity and mainly emphasised the counting of the elements of the set. Thus, for more than half of the preschool teachers, the definition of the key terms used in the activities is not present in the introduction: combination, equal, different combinations. Preschool teachers could illustrate these terms with examples and counter-examples in the form of direct teaching with concrete representation and ongoing checking of children's understanding. Some preschool teachers used mathematical discourse in this way but did not specifically introduce it, and some preschool teachers withdrew from it altogether and were therefore unable to focus on what was essential about the problem. Solving combinatorics problems is not counting, but systematic counting, which is given the opportunity to become systematic by naming the situation.

When using mathematical terms in examples of how to create arrangements or permutations, the preschool teachers used the wrong terminology. They used the term combination instead of arrangement. This was particularly problematic when the children moved from combinations to permutations, making it more difficult to distinguish between them. There were also some errors in the questions when conducting the activity. As many as 7 out of 11 preschool teachers asked mathematically incorrect questions or used incorrect terminology. Table 2 gives some examples of professional errors in the questions asked by the preschool teachers. In all the questions with errors, the children were unsure and had difficulty understanding what the combination was. For example, instead of a combination of colours, they listed the individual colours. Instead of a combination of two fruits they listed the names of the individual fruits. It is also worth mentioning that in ordinary discourse, in Slovene, the word combination represents any possibility of placing, selecting, combining objects, whereas in mathematics the term combination is reserved for a specific concept, it does not include permutations or variations.

Table 2. Examples of errors in the questions.

Preschool teacher's statement/question (we quote their questions verbatim)	Comment
We'll see what colours we have.	The preschool teacher meant the colours in combinations as she directed the children to the sheet on which the combinations were made. The children understood the question as she asked it, so they started to list all the colours.
If a squirrel is wearing a different hat and scarf, what do we call it?	For this question, the preschool teacher expected a combination as an answer.
Which flavour hasn't been chosen yet?	In this question, the preschool teacher actually asked for a combination of flavours.
Nina, which two combinations will you choose?	By this, the preschool teacher meant one combination of two ice-cream flavours out of three flavours.
Are the two plates different?	The preschool teacher was thinking about whether there were two different combinations of fruit on the plate.
Is no picture the same on a plate?	By this, the preschool teacher meant the combination of pictures on the plate.

5.2. Guiding the solution of a combinatorics problem

The following aspects were analysed to guide the solving of the combinatorics task: the form in which combinations or arrangements were introduced to children, the presence of comparisons between the representations of possibilities, the presence of prevention of children's creating incorrect representations or answers, the promotion of systematic combinations or arrangements, the way in which children's incorrect answers were responded to, and the way in which the activity was concluded or summarised.

The majority of the preschool teachers (8 out of 11) worked in a guided group setting (Table 1). When searching for combinations or arrangements, it was observed that 5 preschool teachers had all children searching for combinations or arrangements at the same time, and 5 preschool teachers had all children searching for combinations or arrangements in a sequence (first one child with one example of a combination or arrangement, then the next child with another example of a combination or arrangement, until all children had taken their turn). One preschool teacher used a combination of both approaches: all children forming the combinations or arrangements at the same time at first, and then in the next extension situation in turn. In 8 out of 11 situations, one or a few combinations or arrangements were made by an individual child, rather than all of them. The solution to the task was shared.

In most of the sequential combinations or arrangements, children compared the resulting combinations or arrangements on an ongoing basis (5 out of 11). Overall, children compared the resulting combinations or arrangements in 8 out of

11 activities and in one case of combination formation, the child compared only his/her representations.

The way in which combinations or arrangements were found also influenced whether preschool teachers responded to or prevented children's incorrect answers. In only one case, where the children made the combinations in sequence, were the repeated combinations or arrangements not corrected on the spot. In addition to responding to incorrect answers (identical combinations or arrangements made) in advance, preschool teachers prevented errors by limiting the material (5 out of 11). This means that the children were given as much material as they needed to make all the combinations or arrangements. Thus, anticipatory error prevention was present in 7 out of 11 implementations. In this case, some preschool teachers restricted the children only with the material or with the on-going correction, while some used both.

When looking for combinations or arrangements, it is also important that the preschool teacher teaches the children how to look for the resulting combinations or arrangements systematically when looking for solutions. Combinations or arrangements can be systematically searched for by substitution, fixing one element and changing the others until all possibilities are exhausted. English (1991) calls this strategy of searching for combinations or arrangements the "odometer" strategy and it is considered to be the most advanced problem-solving strategy in combinatorics. Several studies have shown that preschool children are not yet able to systematically find combinations or arrangements by themselves. In our study we found that the preschool teachers did not use the "odometer" strategy, but most of them (9 out of 11) referred to the systematic search for combinations or arrangements at least in one part of the activity management. Of these, 3 preschool teachers referred to the systematicity of the search by sequencing the resulting options in the task in such a way that the systematicity was illustrated by the representation used, but did not explicitly express this to the children. An example of a visual representation of all permutations of three items (three different flavours of ice cream in a cone, order being important), where the final representation of the arrangements is systematically represented, is shown in Figure 3. Figure 2 shows how we may be limited in representing all possibilities systematically by a wrongly chosen representation. Since we have four snowmen on each sheet of paper, we cannot arrange them systematically and review the resulting possibilities when fixing the selected item (all snowmen, given a choice of three colours of hats and three colours of scarves, are 9, which is impossible to display systematically with the chosen representation – 4 "differently dressed snowmen" per sheet). The remaining preschool teachers made only a cursory reference to systematically searching for solutions by asking questions. For example, when looking at the possibility of choosing two colours of balls out of four colours of balls (Figure 5), the preschool teacher asked the children: *"Do we have red with blue yet? What about red with yellow? What about red with green?"* She then fixed the second colour and asked again for all the options. There was a repetition here, as "red and blue" and "blue and red" are the same combination, but the preschool teacher did not pay attention to this aspect. Often, however, the preschool teachers did not take the opportunity to stress the importance of systematicity. For example, when extending the baseline situation, the preschool teacher added a new hat colour. The children could have designed the new possibilities in a systematic way to combine the new hat colour with all the colours of the scarf (Figure 2). In Table 3 we present an example of a dialogue during a problem-solving activity to find a combination of two ice-cream flavours out of four (Figure 1). The dialogue highlights the systematic way in which all the resulting combinations were examined, but this was not supported by a concrete representation, only by questions. The children could also systematically arrange the resulting combinations by placing the jars with ice-cream flavours in the correct sequence according to the colours chosen to appear in the combinations.

Table 3. Encouraging systematic search for combinations with questions (Ice-cream_1).

C	It is the child's turn to make a new combination of a choice of two colours from a choice of four. He looks at the jars in which the combinations have already been made and takes out a green ball.
T	The preschool teacher reminds him that he already has all the combinations, saying, *"I think we're with kiwi."* He points to all the jars where there is already a kiwi (green ball).
C	The child says, *"We already are."*
T	*"We already are, excellent. Let's see if we've used vanilla with everyone."*
C	The child looks at the jars with the resulting combinations and says: *"We're with three."* Then he takes the red ball.
T	*"Well, are you going to combine this...strawberry (red ball) with...?"*
C	He takes a purple ball and says, *"blueberry."*
T	*"Excellent. Now let's see if we have all the combinations."*
C	*"Yes."*
T	*"Why do you think so?"*
C	The child explains, showing the resulting combinations, *"Because it's strawberry and blackberry, because it's kiwi and blackberry, because it's vanilla and kiwi, because it's kiwi and strawberry, because it's blackberry and vanilla, because it's strawberry and vanilla."*

Note. C = child, T = preschool teacher.

The problem of systematicity was also evident in one of the activities (*Dress_3*), where a girl was asked for clothing options (Figure 7). At the end of the activity, the preschool teacher demonstrated on one paper model of a girl all the clothing options for the girl, choosing from the first set two caps, from the second set two colours of T-shirts, and from the third set two colours of skirts and trousers. The preschool teacher then used the model of the girl to show the options for the girl's clothing. When the preschool teacher had designed a new clothing option, the previous one had disappeared. The child found it difficult to remember which option had already been chosen and the order in which the changes of clothes were made on the girl. Even if she fixed one object, e.g. the hat, first and changed the other two (the shirt and the trousers/skirt), this was too much for the child to remember all the resulting options. It would definitely have been better in this case if the preschool teacher had as many pictures of the girls as there are different ways of solving the problem. This way, all the possibilities would be visible to the children, who could then organise them systematically according to the way in which they searched for the possibilities by fixating on the chosen item.

No preschool teacher planned to synthesise the findings or systematically summarise or guide the children towards conclusions in the lesson preparation. Also, in the implementation, most preschool teachers (9 out of 11) did not give children the opportunity to reflect on whether they had found all possible solutions to the task. Usually, the preschool teacher herself summarised in the conclusion that they had found all the possibilities. In only one case did the preschool teacher ask the children if they had found all the possibilities and encourage them to justify it (dialogue in Table 3) and in only one case did the preschool teacher allow the children to conclude for themselves that there were no more possibilities because they had exhausted them all. Each time the children tried again, they were given the option they had already had. They therefore concluded that there were no newer possibilities. In general, most of the implementations (9 out of 11) were loosely formulated in terms of answering the question posed in the initial task (e.g. *"How many combinations are there? How many different fruit plates can we make?"*), or the preschool teachers obtained the children's agreement that they had found all possible combinations or arrangements. We give a few more examples of activity closure.

Examples of closing activities:

1. Activity *Ice-cream_2*: At the end of the activity on making all possible arrangements of three different flavours of ice-cream, the preschool teacher said, *"When we are making combinations, it is important to think about which place to put which ice-cream ball. And if we think about it, we change the order, we get more possible ice creams."* In this case, the preschool teacher's conclusion emphasised the way to find different arrangements rather than the number of arrangements. She used the term combination instead of arrangement.
2. Activity *Balls_1:* At the end of the activity on finding combinations of two colours of balls out of four colours, the preschool teacher said, *"So if we have fewer different colours we can make fewer combinations, but if we have more, we can make more combinations, right?"*). In this case, the preschool teacher concluded by pointing out how the number of colours influences the number of combinations made and tried to get the children to agree to the explanation given.
3. Activity *Dress_4* and activity *Balls_2:* In both cases the activity ended with the children solving the task without any reflection. In the activity *Dress_4*, the preschool teacher said to the children at the end, *"Well, now whoever wants to put these dresses and hats on the snowman and then they can take them home"*. In the activity *Balls_2*, after checking that the children had correctly filled in the table of all possible arrangements of the three colours of the balls in the 3×3 grid, the preschool teacher said to the children *"Then we have done it right"*. There was no justification as to why this was correct.

From these dialogue examples, in addition to the aspects we have highlighted in each case, it is worth pointing out that the vocabulary, the formation of interrogative sentences, is very modest. The preschool teacher makes practically no difference to the child's speaking in the way he or she asks questions. Probably in an attempt to be closer to the child, to be understood, but at the same time he does not take the opportunity to let the child learn the language and mathematical content in a way that is adapted to him, but still in a language that makes such a difference to the child's knowledge that he can distinguish between what he already knows and what he is newly learning.

Another important aspect that we have analysed in detail is the management of dialogue when solving combinatorics problems. Here we were interested in the consistency between the planned and the executed dialogue in terms of the planning of the types of questions and the achievement of the individual merits of the dialogue in the conduct of the activity.

5.3. Conducting dialogue with children

Alexander (2018) lists five qualities that should be reflected in the management of dialogue in the classroom. Dialogue should be collaborative, supportive, purposeful, reciprocal and cumulative. When analysing the observed implementations, we found that all dialogue management was supportive, most was also purposeful (8 out of 11 implementations), with slightly fewer dialogues being cumulative (7 out of 11 implementations) and reciprocal (6 out of 11 implementations). The dialogue virtue that was realised in the smallest proportion was collaborative. We found that dialogue was collaborative in only 5 out of 11 implementations. In one implementation (*Balls_2*) it was conditionally collaborative, as the activity allowed for participation in finding a joint solution, but the preschool teacher did not make use of this in conducting the dialogue. The preschool teacher worked individually with each child to find a common solution. The dialogue was not purposeful (in terms of achieving the objectives of the combinatorics content) in three of the implementations. In two activities (*Dress_4*; *Balls_2*) the focus was on counting the elements of the set, e.g. *"How many snowmen have a red hat? How many snowmen are there?"*. In the activity on dressing (*Dress_2*) the children only fitted the "dressed objects" into a table of all the possibilities (Figure 7). They did not have the opportunity to design any of the clothing combination options themselves.

Reciprocity of dialogue is when the preschool teacher has a longer dialogue with the children. We have decided that a dialogue is reciprocal if at least two questions are asked. So the preschool teacher uses the child's answer for the next question. An example of a reciprocal dialogue where the child started with a question is shown in Table 4. In the *Tower* activity, the children built a tower with three different coloured blocks. They looked for all possible arrangements of the three different colours of cubes in the tower. Below are some dialogues that reflect the particular features of the dialogue.

It is important to point out that, although dialogue involves reciprocity, it also involves most of the characteristics that we have used in the theoretical work to define what dialogue is not. This means that we are not highlighting it as an example of a desirable dialogue with children, but only showing how dialogue could include elements of reciprocity.

Table 4. Example of a reciprocal dialogue (Tower).

C_1_2	The children work in pairs to make their own column of 3 colours of cubes.
T	*"But do you think it will be OK?"*
C_1_2	The two children argue a bit about how it should be. *"No, that's not right."*
T	The preschool teacher intervenes and says, *"Well, which colour are you going to start with?"*
C_1	*"With blue."*
T	*"With blue, but do we have blue down here somewhere?"*
C_1	*"No."*
T	*"No, well, then it doesn't matter which cube we put on top. So we have a third tower. Agreed?"*
C_1_2	*"Yes."*
T	*"Very good."*

Note. C_1 = child 1, C_1_2 = child 1 and child 2, T = preschool teacher.

Collaborative dialogue means that the children co-create the solution to the task through the conversation. The preschool teacher encourages the children to exchange ideas about the solution to the task. An example of collaborative dialogue is shown in Table 5. The dialogue took place while the children were asked to create four different choices of a bottom clothing to go with the top clothing (*Dress_5*, Figure 8). They had to choose between two colours of shirts and two colours of trousers.

Table 5. Example of a collaborative dialogue (Dress_5).

T	In a situation where they have two identical combinations, the preschool teacher asks, *"How do we switch?"*
C_1	The first child makes a suggestion: *"I know."* He takes the yellow T-shirt from the non-repeating combination and puts it with the blue trousers. In this way, they recreate the combination they already have, but in a different place.
T	*"Is it OK?"*
C_2	The other child chimes in, *"No, but now these two are the same."* She points to the same two combinations.
T	*"What can we do now?"*
C_3	A third child chimes in with a new suggestion: *"Then we can change our trousers."*
T	*"How? Show me."*
C_4	A fourth child comes in and takes out a green T-shirt and puts on a pair of red trousers to make the last combination they were missing.

C = child 1, T = preschool teacher.

Figure 8. Formatting the option to select the lower part of the clothing with the upper part (Dress_5).

Table 6. Example of a dialogue with multiple features: purposeful, collaborative, cumulative (Tower).

T	*"So which combination are we left with now? But maybe someone knows?"*
C_1	The child points to the colour arrangement, already created and says: *"Yes, yes, that one."*
T	*"Yes, but how do the colours go in order from bottom to top, but maybe someone knows?"*
C_3	The next child joins in and says: *"I know I can do blue, white and red."*
T	*"Blue, white and red. Well, let's see. This one below is blue...blue, white and red. This one is blue, red and white. We already have this combination* (shows the towers where the blue is below and tells the sequence of colours). *That is, it will not be like this. What will it be? Let's look at the colours below. Here below is blue* (put the towers together), *here is red* (put the towers together), and *here is white* (only one tower where white is below). *So which colour will be below ?"*
C_4	*"White."*
T	*"White, how are we going to put it up then?"*
C_5	The child suggests: *"That it will be blue here now."*
C_6	The other child corrects him, *"It will be red."*
C_7	The next child comments, *"No, because they will be the same."*
C_2	*"I'll try, that's how."* In doing so, arrange the colours in the column accordingly.
T	*"Well, let's see, what do we have, aren't they different?"*
C_ALL	*"Yeah."*
T	*"And how many combinations did we get?"*
C_1	*"Six."*

C = child, T = preschool teacher.

The dialogue in Table 6 (*Tower*) is an example of a cumulative dialogue, as the children worked with the preschool teacher to build on mathematical ideas and to make a tower of three different coloured cubes in arrangements they did not already have. The dialogue is also purposeful as it has a clear goal, i.e. to find the last possible arrangement of the three colours in the tower. It is also collaborative, as several children contribute their ideas to the joint solution of the task. Seven children participated in the dialogue to find a common solution. The preschool teacher used the term combination incorrectly in the discussion, as the children were looking for different arrangements of colours, not combinations. The dialogue also reflects the encouragement to systematically search for arrangements of the three elements.

The following example of dialogue, presented in Table 7, is also purposeful, supportive, cumulative and collaborative. The dialogue was led by the preschool teacher with the aim of finding all the different combinations of dishes (*Lunch* activity), choosing between meat and fish and between three different side dishes: pasta, rice or potatoes. The children first created their own dish options by placing the chosen pictures of the dishes on a plate, then looked together to see if they had found all the options. The dialogue also reflects the reference to the systematic way of finding combinations.

Table 7. Example of a dialogue that is purposeful, supportive, cumulative and collaborative (Lunch).

T	*"Let's see what you have done. What have you put together? You put the fish next to the macaroni. That is to say, we already have this combination. What else can we put with the fish? Meat and fish do not go together. You can put potatoes, rice or pasta as a side dish."*
C_1	*"We'll put meat with the fish."*
T	*"No, fish and meat don't go together."*
C_2	During this time, the girl makes a new combination of fish and rice.
T	*"Great, now we have fish and macaroni, fish and potatoes, and fish and rice. What else are we going to put? Now we have meat on our plate, what else is missing?"*
C_2	The girl puts the potatoes on one plate next to the meat and the meat on the other.
T	*"Well done C2. Now we have meat and potatoes, meat and macaroni on the plate. What's left?"*
C_3	*"Rice."*
T	*"So meat and rice. Well done".*

C = child, T = preschool teacher.

However, it can again be concluded from the preschool teacher's questions in Table 7 that it cannot be disputed that all the features of dialogue are included in terms of their definitions, but there is very little information about the content itself and the ways in which children's thinking is stimulated. Again, it could be concluded that the preschool teacher is not opening up the children's ignorance, but is constantly striving to ensure that the child does not over-exert himself in his thinking. It is quick to cheer up the right solution and praise the children, but it does not make the children doubt and reconsider.

In addition to the strengths of dialogue management, we were also interested in the questions that preschool teachers ask during the combinatorics activities and whether they allow children to represent their ideas. We only considered questions that related to the content of the combinatorics activity (e.g. we excluded questions to learn about the set, the type and number of elements to choose from and combine, such as *“How many different colours of hats do we have? How many types of fruit do we have? Which fruit do we have?”*). We found that all the activities were designed in such a way that the children had the opportunity to represent their ideas. Examples of questions to represent children’s ideas: *“Tarik, can you show here on the plate what combination Tara has chosen. How can we dress Meta for the first day? Can you show.”* We further observed the proportions of each category of questions asked by the preschool teachers. We created three categories of questions: describing questions, reasoning questions and questions or situations extending the initial task. In all implementations, the highest proportion of questions asked by preschool teachers was describing. The proportion of description questions ranged between 66 % and 100 % (minimum of 4 and maximum of 35 questions) of all questions asked in the activities. Examples of description questions for the three different combinatorial situations are presented in Table 8.

Table 8. Examples of description questions in three different contexts.

Combinatorial situation	**Examples of description questions**
Build columns of cubes of three different colours (*Tower*).	*How many combinations did we get?* *But do we have any towers that have white at the bottom?* *What do you think we will be able to change, now there is white, red, blue?* *Which colour will they start with?* *But are all the towers different?*
Choose two fruits from a choice of four (*Fruits*).	*What other fruit plates could you make with the fruit that’s left?* *Jan, what else could you combine, which fruit?* *Which combinations have we not used yet?* *How could we change the fruit plate to make it like nothing we have before?* *Which type of fruit would you change so that the plate is not the same?* *Which fruit have we already combined with blueberries?* *How many different fruit plates can be made with two different fruits?*
Dressing a snowman with a hat and scarf (*Dress_1*).	*Do you know what it means that no two snowmen can have the same combination?* *Is there another snowman like yours? Katarina, see if these are now different combinations.* *What could Ema replace?*

In 4 of the 11 implementations, justification questions were also asked (minimum 1 question and maximum 4 questions). Examples of justification questions: *“Why do you think so? Why did you make this decision? Why do you think Tara that we have already used this combination? Why is this not right?”*

8 out of 11 preschool teachers included at least one extension question. The most common extensions included increasing the number of elements of a set or asking how to increase the number of different options. There was also one example of extension in terms of combinations with repetition (ice-cream flavours can be repeated, previously only different flavours were combined, *Ice-cream_2*), one example of reducing the number of elements of the set (*Balls_2*) and moving to a new

structure of the combinatorial situation (*Ice-cream_2*, *Dress_4*, *Lunch*, *Dress_5*). Examples of extension questions for a few activities are presented below.

Activity *Balls_1*: *"But do you think that now we've added purple, there will be more combinations?"*

Activity *Dress_1*: *"And if I don't have these caps now, what can I do? What else could I add to give 12 friends?"*

Activity *Dress_3*: *"Dress Mia again for three days if we add one more colour to the cap."*

Activity *Ice-cream_2*: *"You can make the ice-cream as before, or you can use the same flavour twice, and then we'll count how many combinations we get."*

The last aspect we have analysed is the consistency between the design of the questions in the preschool teacher's lesson preparation and the use of these questions to guide the solution of the combinatorics task. We assumed that question planning is consistent with implementation when at least half of the questions are consistent. We found that in only one implementation was the question planning consistent with the implementation. In this case, the preschool teacher asked more than half of the questions she had planned and did not add new questions. In 9 out of 11 implementations, more different questions were asked in the implementation than in the planning. In only one implementation was the reverse the case, the preschool teacher planned more different questions than she used in the implementation. The results show that most preschool teachers adapted to the children's answers while working with them, adding new questions that they had not foreseen in the planning phase. Based on the questions presented in our analysis, it is very easy to see that most of the questions were unplanned.

In terms of issues, there were also more professional errors in implementation than in design. Mathematical errors were present in 7 implementations and in 4 cases of activity planning. The preschool teachers who made professional errors in planning also made them in implementation. This suggests that some of the errors can be prevented by careful scrutiny of the teaching preparation and appropriate feedback to the preschool teachers. In the planning and implementation, some preschool teachers were also observed to use the question *"Is it correct?"* which is inappropriate in terms of determining the appropriateness of combinations or arrangements. The question is not whether a combination or arrangement is correct, but whether it is different from the previous one. It is not difficult to summarise that such a question is a consequence of understanding mathematics in such a way that we are dealing with either a correct or an incorrect solution. Yet there is too little awareness among preschool teachers of the promotion of procedural thinking in children.

We would also like to highlight two other categories of questions that reduced the quality of the guided dialogues when solving the combinatorics task. The first category is questions that distract children from the point. Examples of such questions are: *"How many snowmen have an orange hat? How many snowmen are wearing green clothes? How many snowmen would you have if you had 2 more snowmen, how about 3 more snowmen?"* The second category is questions that are not based on something tangible. This type of question was found in a large proportion of the implementations (7 out of 11). These questions give the impression that the child does not understand them, but in fact they are questions that the child cannot answer at all. Examples of such questions are: *"How many friends do I have? Do I have enough hats and scarves? Is there anything else I could do to give to 12 friends? What would you do if two more guests came? If 6 friends came, would they all get their lunch?"* The use of this type of question is also illustrated in Table 9, which presents part of the dialogue during the activity *Dress_3*. At the beginning of the dialogue in this activity, the preschool teacher asked the children to show a picture of a girl and how she could dress her up to go to school. They chose between two different tops and three different bottoms. They had to take into account that the girl would be dressed differently every day.

Table 9. Example of a part of dialogue management with a question that is not based on something tangible (Dress_3).

T	*"And we have a hip of T-shirts?"*
C	*"We have two."*
T	*"Two, how many times does Meta go to school?"*
C	*"Three times."*
T	*"Three times, that is...what does that mean?"*

C = child, T = preschool teacher.

Another example of dialogue management where the preschool teacher asks questions that are not based on something tangible is presented in Table 10 (*Dress_1*). In the *Dress_1* activity, the preschool teacher had pictures of twelve snowmen (one for each friend, Figure 2) on the table. She thought that the children would connect the answer to the question about how many friends she had with the number of snowmen she had prepared and count them.

Table 10. Example of a part of the dialogue management with a question that is not based on something tangible (Dress_1).

T	*"I decided to give snowmen to my friends. Each friend will get one snowman. How many friends do you think I have? Does anyone know how many I have?"*
C	The children guess: "Six."
T	*"Six, why do you think I have six friends? Does anyone have any other idea how many friends I have? If everyone gets one snowman* (pointing to snowmen on a piece of paper, he has three sheets, four snowmen on each sheet). *How would that tell me how many friends I have?"*

C = child, T = preschool teacher.

6. Conclusions

As we mentioned in the introduction, in the Slovenian context (and this is not only a characteristic of the Slovenian context), most preschool teachers advocate the so-called constructivist approach, i.e. the child's own construction of knowledge, and are opposed to direct teaching. If teaching is to be modern, the preschool teacher should be as little as possible in the teaching role, should withdraw as a mediator between the material and the children, or should only get involved when the children encounter difficulties. The analysis of the videos showed that the preschool teachers acted as dictated by modern guidelines: they used concrete material, the children were in groups, they had a dialogue with the children, they did not directly teach the children. If we had not analysed the videos in depth to see where the preschool teachers fail, we might believe that the kindergarten is working well: the children are having new experiences, the preschool teacher is supporting them in their learning, talking to them... And where do they fail the most? In not giving children new knowledge in the way they could. Before we argue our case, we will briefly look at the essential emphases of the role of the preschool teacher in the theory of constructivism (it is practically impossible to exhaust these epistemological theories at this point).

Despite conceptual differences in the understanding of constructivism, proponents of constructivist theory generally agree that (1) all knowledge is constructed, (2) there are cognitive structures that are activated in the processes of construction, (3) cognitive structures are constantly evolving (intentional activity causes transformation of existing structures), and (4) the acceptance of constructivism as a cognitive theory leads to the acceptance of didactic constructivism (Noddings, 1990). The theory of constructivism, which, as stated, is essentially an epistemological theory (Glasersfeld, 1984), is based on two key premises: (1) the cognitive activity of the individual aims at establishing laws in the experiential world, and (2) the establishment of laws presupposes experiences that are continuously judged by the individual in terms of equivalence and individual identity (das gleiche, dasselbe; stesso, medesimo). This conceptual starting point is extremely general and non-prescriptive (it is not a "guide" for pedagogical practice). These two key points do not offer any unique derivations for the preschool teacher's teaching. The theory of constructivism implies the importance of the preschool teacher as a transmitter of knowledge. The preschool teacher has to think carefully about how to guide the child to construct knowledge.

Although the teacher in the "modern" (as opposed to the "traditional") understanding of teaching and learning are systematically being removed from their teaching role, and there is still a concern in the field of preschool education to "teach the children a lesson", we have decided to highlight in our research the importance of the preschool teacher's role in the kindergarten. We argue that the preschool teacher, as a knowledge holder, has a key role to play in the teaching of mathematics in order for children to progress in their knowledge. The preschool teacher must manage the processes of knowledge acquisition in a very deliberate way, so as to maintain a reference to previous knowledge when building on it, that is to say, to maintain equivalence, which at the same time introduces difference, which is again new knowledge for the child, and the relation between the same and the different must be such that the difference is not too big for the child and would lead to incomprehension, and not too small, because it would not lead to new knowledge (Krek and Hodnik, 2022). As already mentioned the theory of constructivism is an epistemological theory, which by definition does not provide instructions for teaching but prompts us to use appropriate approaches to build on knowledge, whereby the preschool teacher *continuously opens ignorance* in the learners (the realisation that they do not yet have this knowledge).

We want to stress this because constructivism in Slovenia in general (this is not to say that this is not the case also elsewhere) is understood in a rather simplistic sense, in the sense of "to construct", which refers to the activity of "doing" with the material, in a group, in an interaction, rather than to *the activity of thinking*.

In our study we have found out that:

- although the preschool teachers advocate "constructivist practice", the implementation of the kindergarten activities we analysed *does not reflect this* (guiding the children in the use of materials, working with each one individually even though the appearance of working as a group, demonstration of dialogue),
- the preschool teachers *did not demonstrate their role as knowledge holder* (from the questions posed they are just a little above children's);
- the preschool teachers know the immediate goal, the context, the choice of the material, but *not how to stimulate thinking*!

Therefore, the consequence we can draw would be *that they should follow constructivist theory* in the sense we have described it. They have yet to acquire the knowledge of how *as knowledge holders* to lead the dialogue *to stimulate thinking*.

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Vzgojiteljevo načrtovanje in izvedba dialoškega poučevanja pri obravnavi kombinatoričnih situacij

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Povzetek. Predšolsko obdobje za otroka, še posebej če obiskuje vrtec, predstavlja vstop v sistematično pridobivanje matematičnih pojmov. Ti so v tem obdobju vezani na materialno realnost, v nadaljnjem učenju pa se od nje pojmi vedno bolj odmikajo, pojmi postajajo abstraktni miselni objekti. Poleg nekaterih najbolj pogosto obravnavanih vsebin v predšolskem obdobju (na primer štetje, geometrijske oblike) raziskave s področja zgodnjega učenja matematike poudarjajo pomen učenja o preprostih kombinatoričnih situacijah. Pojmi iz kombinatorike, če so ustrezno reprezentirani in osmišljeni, podpirajo otrokov razvoj sposobnosti posploševanja in sistematičnega mišljenja, hkrati so precej samostojne vsebine, torej niso neposredno vezane na znanje otroka o drugih matematičnih vsebinah čeprav se učitelja in vzgojitelja v "sodobnem" pojmovanju (za razliko od "tradicionalnega") poučevanja in učenja sistematično umika iz njegove poučevalne vloge, na področju predšolske vzgoje pa pri nas obstaja še skrb, da bi otroke "pošolali", smo se odločili izpostaviti oz. ponovno osvežiti pomen vzgojiteljevega delovanja v vrtcu. V prispevku zavzemamo stališče, da ima vzgojitelj, kot nosilec znanja, pri poučevanju matematike ključno vlogo pri tem, da otrok napreduje v znanju. Ena od odlik dobrega poučevanja je premišljeno načrtovanje in izvedba dialoga z otroki v okviru dialoškega poučevanja, ki praktično nikoli ne nastopa samostojno, prepleta se z direktnim poučevanjem – vzgojitelj pri delu namreč določena znanja, termine, postopke ipd. tudi neposredno posreduje. V prispevku nas zanima, kako vzgojitelj načrtuje in izvaja dialog s predšolskimi otroki pri obravnavi vsebin iz preprostih kombinatoričnih situacij. Vzgojiteljev dialog bomo proučevali z vidika vrste in kakovosti vprašanj ter z vidika vzgojiteljeve poučevalne vloge. V vzorec bomo vključili skupino vzgojiteljev, ki so v okviru predmeta o zgodnjem učenju matematike pridobili ustrezna teoretična znanja o vodenju dialoga s predšolskimi otroki, na osnovi katerega so načrtovali in izvedli aktivnosti v vrtcu. Vpogled v kakovost vzgojiteljevega načrtovanja in izvajanja matematičnega dialoga bomo dobili s kodiranjem in kvalitativno analizo transkribiranih posnetkov vzgojiteljevega vodenja dialoga z otroki v procesu izvajanja matematičnih aktivnosti ter s primerjanjem načrtovanja vodenja dialoga ter njegovo izvedbo. Doprinos našega prispevka vidimo v tem, da za razliko od raziskav, ki v glavnem proučujejo otrokovo razmišljanje pri reševanju izbranih problemov iz kombinatorike, naša raziskava izpostavlja vlogo vzgojitelja kot bistvenega akterja v procesu otrokovega učenja matematike. Nadalje bomo v prispevku predstavili tudi primere kakovostnih dialogov: tako z vidika načina izvajanja kot z matematičnega vidika.

Ključne besede: kombinatorika, matematika, predšolski otrok, dialoško poučevanje, vzgojitelj

The Importance of the Mathematics Teacher and Special Educator Working Together to Support Low-Achieving Pupils in Mathematics

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Abstract. The number of pupils receiving additional professional support in learning mathematics is increasing every year. Ensuring quality support for lower-achieving pupils is a growing concern in mathematics education, including the issue of how to provide competent professionals to provide additional professional support. Mathematics at the secondary level is the subject with the highest number of lessons after Slovene language, is complex, abstract and increasingly less related to material reality. At this level, it is no longer possible to strictly separate the overcoming of pupils' learning difficulties in individual mathematical topics from the teaching support, which is why it is essential to have the effective collaboration of different professionals who bring competences and knowledge from their own areas of expertise to the learning process. Currently, individualised intervention as additional professional support in mathematics outside the classroom is the predominant approach in Slovenian elementary schools, but experts point out that in order to achieve optimal learning performance as well as social inclusion, more support should be provided within the classroom (Pulec Lah & Košir, 2015). Some Slovenian schools are starting this kind of practice, which requires a lot of teamwork between the secondary mathematics teacher and the provider of additional professional support, as well as effective organisation. The aim of our research was to investigate in what form the selected schools implement the support, in what way, in what form and how often the special educator and the secondary mathematics teacher collaborate, in which cases, according to the professionals, individual or inclusive implementation is more appropriate, and what are the advantages and disadvantages of these two methods. According to the Rules on educational qualification for teachers and other education staff in the basic school education programme (2022), different profiles of professional staff provide additional professional support, so our aim was to find out how the profile of the professional staff member providing each form of additional support is determined in the selected school. Through in-depth analysis of systematic observation of individual and inclusive mathematics teaching and interviews with mathematics teachers and special educators, we found that effective collaboration between the mathematics teacher and the special educator is key to achieving the learning goals in mathematics teaching. Quality delivery of mathematics teaching to low-achieving pupils requires quality teamwork between all those involved. In conclusion, we suggest that mathematics teachers and special educators should work together to plan mathematics lessons in the case of inclusive implementation, especially for more challenging content. If the special educator is providing individual support to the pupil, we suggest that in inclusive lessons, the special educator should be present during the mathematics teacher's explanation and should keep a record in the form prepared for this purpose of key mathematical vocabulary, representations of concepts and procedures, problem chaining and other relevant guidance that can be used when working with the pupil outside the classroom in order to ensure consistency in the treatment of the concept. This is a prerequisite for the special educator to establish a dialogue with the pupils, which is further developed in accordance with the special education profession. When a pupil receives a disjointed professional and special didactic way of support, the results are significantly worse, despite the efforts of the educational staff, than when the mathematics teacher and the special educator work closely together in a pupil-centred way.

Keywords: special educator, secondary mathematics teacher, additional professional support, co-teaching, inclusion

1. Introduction

The number of pupils referred to special education programmes with adapted provision and additional professional support is increasing every year (*Data on pupils with special needs in elementary schools with adapted provision and additional support, n.d.*). In Slovenia, pupils with learning difficulties in mathematics are supported in primary school within a five-tier model of support based on the *response-to-intervention* model (Kavkler, 2011). The intensity of support is escalated from the first to the fifth level, with the fifth level (if the pupil is not enrolled in school with a lower educational standard or school with a modified programme and an equivalent educational standard) representing the orientation to a modified programme in form of additional professional support (APS), where the pupil is allocated a certain number of hours of additional support. The purpose of the additional support is to ensure equal opportunities or to enable the pupil to achieve his learning objectives and to progress in accordance with his abilities, despite his deficits. Low achieving pupils receive help with overcoming learning difficulties in the form of overcoming deficits, barriers or disabilities and learning support as part of APS (Placement of children with special needs act, 2011). Overcoming deficits, barriers or disabilities is help provided by a special educator who helps the child to overcome barriers and empowers the child to learn (Placement of children with special needs act, 2011). A special educator is a teacher without a subject specialisation who specialises in working with children with disabilities, including learning difficulties in mathematics. Learning support is related to assistance in learning mathematics content, and the provider of mathematical content is usually the mathematics teacher (Opara, 2015). Pupils with learning difficulties in mathematics face various barriers to achieving their learning goals in mathematics. Barriers may include maintaining persistence in learning, vocabulary difficulties, difficulties with logical thinking and reasoning, and memory difficulties (Rexroat-Frazier & Chamberlin, 2019).

In the following, we will focus on the collaboration between the special educator and the mathematics teacher in overcoming learning difficulties in mathematics at secondary school level, as the mathematics content in grades 6, 7, 8 and 9 (pupils aged from 11 to 15) is complex and more abstract than at primary school level.

2. Implementation of APS for overcoming pupil's difficulties in mathematics

A pupil's success in mathematics is closely linked to his mathematics teacher. In addition to the teacher's characteristics, the pupil's success is strongly influenced by the teacher's motivation, approach, attitude towards teaching, subject expertise, and pedagogical and curriculum knowledge. It is also very important that the teacher understands the mathematical content, as only then can he interpret it in a way that is relevant and understandable to the pupils and divide it into meaningful smaller units, thus also helping to create the thought processes that enable pupils to understand the mathematical content. It is therefore essential to ensure that learners with learning difficulties are provided with the right profile of educator to equip them to achieve their goals (Barmby et al., 2010).

Rules on educational qualification for teachers and other education staff in the basic school education programme (2022), article 83 states that additional professional support such as overcoming deficits, barriers or disabilities and learning support in mathematics may be provided by a defectologist, pedagogue, psychologist, social pedagogue, speech therapist, special and rehabilitation pedagogue, or inclusive pedagogue. For all these professional profiles, we use the term special educator. At the same time, mathematics learning support may be provided by mathematics teachers who have completed a further training programme for teachers of additional professional support (Rules on educational qualification for teachers and other education staff in the basic school education programme, 2022). Opara (2015) points out certain shortcomings of the provisions: neither pedagogues nor psychologists acquire the skills and competences to work directly with pupils with learning difficulties as part of their studies, whereas special pedagogues and inclusive pedagogues are trained in this area. At the same time, Opara et al. (2010) and Opara (2015) point out that in practice, at the secondary school level, APS in the form of overcoming deficits, barriers or disabilities is predominantly implemented in the form of learning support. Thus, learning support, which is mainly about additional explanations and reinforcement, is largely provided by special educators instead of focusing on re-education, rehabilitation and deficit compensation in the context of overcoming deficits, barriers and disabilities (Vršnik Perše et al., 2016; Opara, 2015) As a result, special, inclusive, social pedagogues and psychologists are often faced with the challenges of interpreting mathematical content at the secondary school level, and consequently spend less time providing special educational support (Vršnik Perše et al., 2016).

Of all the profiles of special educators mentioned above, only special rehabilitation pedagogues have one subject within their studies specifically focused on the acquisition of knowledge in learning and teaching mathematics, but mainly related to the content covered in lower-education primary schools (Special and rehabilitation pedagogy study programme, University of Ljubljana Faculty of Education[1]). In addition, students on this course also study the Characteristics and Assessment of People with Learning Disabilities and the Strategies for Working with Students with Learning Disabilities strand in relation to learning difficulties in mathematics. The other profiles do not have any mathematics-related subjects in the study programmes related to the professionals working in the area of APS, for example, study programmes of the University of Ljubljana[2]: Master's degree programme, Inclusive Pedagogy, Special and Rehabilitation Pedagogy (all Faculty of Education), Psychology (Faculty of Arts). In the case of APS implemented outside the classroom, we are thus faced with the problem that no educator has both the special-pedagogical and the mathematical-didactic knowledge and skills that are indispensable for the integrated help and support of pupils with learning difficulties.

Mathematical content at the secondary school level is broad, complex, abstract, and less connected to material reality than at the primary school level, so it is no longer possible to strictly separate overcoming deficits, barriers, or disabilities from learning support, as deficits are overcome through content in a mathematical context. Teaching and learning strategies are also done through the mathematical content and in relation to the learning topic in which the learner is struggling, so the strategies cannot be taught in isolation (Van Garderen et al., 2013). Therefore, to successfully help a learner to overcome deficits, barriers or disabilities, the APS provider not only needs to know the mathematical content, but also needs to be qualified in the field of mathematics education. In similar opinion Flores et al. (2010), following the US Classroom of Education's 'No Child Left Behind' reform, shows that in the United States, special educators also need to be highly qualified in mathematics teaching in order to provide more holistic help and support.

An effective educational process must therefore be based on a multidisciplinary approach, where both teachers and special educators combine their professional pedagogical skills and competences. In this way, adapting the teaching process to the needs of the child is more effective (Snyder, 1999; Tapasak & Walther-Thomas, 1999; Tichenor et al., 1998). In order to strengthen the inclusive paradigm in schools, Buli-Holmberg and Jeyaprathaban (2016) point out that the success of a particular APS implementation needs to be judged and built upon in relation to the criteria for the effectiveness of inclusive practice, which encompasses the domains of interactions, supports and adaptations.

Inclusion, based on the right of children to be included in schools and the fact that inclusive education is more effective than segregated schooling, is complex to implement, requiring all professionals to identify and consider the needs of pupils

[1] Znanju dajemo besedo | Pedagoška fakulteta (`uni-lj.si`)
[2] `https://www.uni-lj.si/eng/`

and make appropriate adjustments (Vršnik Perše et al., 2016). In order to achieve mastery of learning, pupils with learning disabilities also need several adaptations in adjusting classroom equipment, learning materials and delivery of instruction. In the area of interactions, it is necessary to assess how much collaboration is required by the desired form of APS delivery, both between the teacher, special educator and the pupil, and between the pupil and his peers. At the same time, it is necessary to critically assess how much support each form of APS provides and what kind of support the pupil needs. Pupils with learning difficulties in mathematics need more support from the mathematics teacher in order to achieve mastery of learning, while support from special educators is needed for the specific help and adaptations to teaching and activities that they provide. However, support in the school environment is not only needed from professionals, but also from peers who represent the pupil's social environment in school (Buli-Holmberg & Jeyaprathaban, 2016).

Regardless of the form of the additional support in mathematics, its success depends on the teamwork of the mathematics teacher and the special educator. Their collaboration starts as soon as the teacher notices that a pupil is experiencing difficulties that cannot be overcome with good teaching practice. The teacher and the special educator then work together in a process of support throughout the elementary school to enable the learner to achieve his/her goals through adjustments to the curriculum. Their role is to plan, implement and evaluate support and assistance as a team (Friend & Bursuck, 2002).

Successful teamwork requires the collaboration of all team members, with an emphasis on the collaboration of the special educator and the maths teacher, as their expertise differs considerably. The success of the collaboration depends to a large extent on the willingness to create a suitable environment and plan for the collaboration. In the process of planning teamwork, the mathematics teacher and the special educator should clearly divide roles, set responsibilities and goals, set expectations, and discuss possible challenges to the collaboration. This is the only way to create an appropriate environment for collaboration, which is a prerequisite for quality pupil support (Murawski & Dieker, 2004).

The Placement of children with special needs act (2011) states that APS can be provided either within or outside the classroom. Several studies (Caf, 2015; Kobolt & Rapuš-Pavel, 2009; Vršnik Perše, 2005) indicate that in elementary schools at secondary school level, APS is most often provided on an individual basis outside the classroom. APS is also relatively often carried out individually in the classroom, and most rarely in groups in the classroom. There is no clear answer as to whether it is better to provide APS inside or outside the classroom. Opara (2015) points out that each form has its advantages and disadvantages, which need to be assessed according to the needs of the pupil.

In the following, we will identify the importance of close collaboration between the professionals involved in the implementation of providing APS in mathematics outside or inside the classroom. We will refer to the information exchanged between team members, the planning of the collaboration and the forms of collaboration in the planning, implementation, and evaluation phases of providing APS.

2.1. Special educator and maths teacher collaboration during APS outside the classroom

Individual delivery of APS outside the classroom allows the pupil to be more actively involved in the planning, monitoring, and evaluation of the learning process and to achieve the goals more effectively than providing support within the classroom (Choate, 2000, in Buli-Holmberg and Jeyaprathaban, 2016). In addition, pupils more easily develop an awareness of their areas of weakness and strength (Brinckerhoff, 1994; Durlak et al., 1994; Field, 1996; Scanlon & Mellard, 2002). In individual work, the pace of work and content can be more easily adapted to the needs of the individual. By providing APS outside of the classroom, it is easier to identify the learner's level of knowledge and the nature of his/her learning difficulties and to adapt the content of the treatment accordingly. Despite some advantages, the main challenge is how to ensure the social inclusion and presence of the pupil during the mathematics teacher's delivery of the new learning content. Moreover, the individualised form of APS outside the classroom does not fully meet the criteria of inclusion in education, as it deprives pupils of the opportunity to interact with their peers (Buli-Holmberg & Jeyaprathaban, 2016). Expert groups in Slovenian schools also find that the provision of individualised forms of extra-curricular support outside the classroom has the greatest impact on learning achievement, but significantly less on social inclusion. Vršnik Perše et al. (2016) thus recommend that other forms of additional support should be strengthened.

In practice, there are problems with the lack of information flow about pupils with learning difficulties between special educators and maths teachers, which has a negative impact on the quality of planning, implementation, and evaluation of APS outside the classroom. There are no provisions specifying what information should be exchanged between the teacher and the special educator in the chosen subject. Thus, in many schools, there is no adequate flow of information between teachers and special educators, as this is left to the personality traits and self-initiative of both, which often results in very infrequent and reduced communication (Vršnik Perše et al., 2016).

In both domestic and foreign education systems, the main challenge that hinders practitioners from teamwork is the lack of time and coordination (Fennick & Liddy, 2001; Friend et al., 2010; Sileo & van Garderen, 2010). As school breaks are too short for team planning and evaluation, Opara (2015) suggests that a viable solution is to establish an organisational form of collaboration that would take place at least once a week. Barriers to collaboration also arise from the divergence of teacher and special education demands on the pupil, the assignment of different importance to different activities based on personal preferences, uneven division of labour, mutual adjustment and personality mismatches (Vršnik Perše et al., 2016). Barriers faced by teaching staff include lack of support at administrative level and lack of knowledge (Baker & Zigmond, 1995; King & Youngs, 2003; Scruggs et al., 2007).

Special educators cite the different expertise of special educators and mathematics teachers as a common problem that hinders teamwork (Ducman, 2012). Knowledge from different areas of expertise should enrich the collaborative relationship, but in the study by Ducman (2012), special educators state that due to a lack of knowledge, teachers do not show enough understanding and interest in collaborating to teach pupils with learning difficulties. Collaboration between special educators and mathematics teachers is important in many aspects. One of those that we would like to highlight is the integration of the expertise of the special educator and the mathematics teacher, which is a prerequisite for an appropriate holistic approach. This requires more time, effort and additional training and is conditioned by the self-initiative and flexibility of the individuals (Ducman, 2012).

2.2. Special educator and maths teacher collaboration during APS within the classroom

In addition to the implementation of APS outside the classroom, the implementation of APS inside the classroom in the form of co-teaching is increasingly being promoted. In this form, the mathematics teacher and the special educator work directly together within the mathematics lesson, where their roles can be different (Vršnik Perše et al., 2016).

Since the goal of inclusive teaching is to educate a diverse group of learners, co-teaching is a way of bringing together and integrating different professionals involved (Moorehead & Grillo, 2013). Collaboration between a special educator and a mathematics teacher is therefore a prerequisite for true inclusive practice, as it makes the best use of the knowledge, personal characteristics and skills of all the educators involved (Buli-Holmberg & Jeyaprathaban, 2016).

The school management has a very strong influence on the success and implementation of co-teaching. According to Nagode and Bregar-Golobič (2008), it is the principal of the school who, with his/her positive attitude towards the inclusion of pupils with special needs, enforces the norms and standards for different forms of work with them. The principal must be careful to adapt changes in the forms of additional support to teamwork gradually, considering the opinions and suggestions of all those involved. He/she should also present an appropriate cooperation plan, provide support and plan time for the cooperation of the mathematics teacher and the special educator (Murawski & Dieker, 2004).

One of the conditions for successful teamwork planning is the willingness of the maths teacher and the special educator to work together. If the school leadership also contributes to supporting the planning of time for co-teaching, the co-teaching practitioners are provided with an appropriate environment for planning (Pratt et al., 2016).

When practitioners decide to implement co-teaching, the first thing to do is to divide roles and responsibilities. As stated by Watt et al. (2013) and Rexroat-Frazier and Chamberlin (2019), the latter is essential for the development of close collaboration, as it is the only way to avoid potential confusion with the perception of roles in co-teaching and potential conflicts. In mathematics in particular, the division of roles is considerably more challenging, as the specificity of the subject leads to large gaps between the skills and competences of the mathematics teacher and the special educator (Moorehead & Grillo, 2013). In a collaboration between a mathematics teacher and a special educator, the mathematics teacher with mathematics-didactic skills usually takes the leadership role. Care must be taken to ensure that the special educator agrees to this division of roles. Murawski and Dieker (2004) suggest that for successful collaboration, all in the team should discuss goals, aspirations, and personal preferences before planning. It is also important that they get to know each other well and get to know each other's way of working, the mathematics curriculum, and individual adaptations, and familiarise themselves with previous work with pupils who need APS. It is also crucial that they agree on the ways and frequency of evaluating both the pupil's progress and the relevance and effectiveness of the collaboration (Murawski & Dieker, 2004).

In the continuation we are presenting six different forms of co-teaching according to Friend et al. (2010), within which the special educator and the mathematics teacher can play different roles.

1. **One teaches, one observes** is a form in which the special educator is present in the classroom for the sole purpose of observing and collecting the pupil's cognitive, behavioural, social and other characteristics. It is the only one of the formats that does not require collaboration between the special educator and the mathematics teacher, but it is an appropriate way for the special educator to learn about mathematical content and the way the mathematics teacher works (Carty & Marie Farrell, 2018).
2. **One teaches, one assists** is a form in which the maths teacher takes the lead and the special educator provides ongoing support and help to pupils who need it. Because this type of support is not limited to the pupils who need extra help, it can be distributed among all pupils by the special educator. Due to the specific nature of mathematics, the help and support provided by the special educator is mainly in the form of additional instructions, short additional explanations and repetitions of what the teacher has said. In practice, this is the most common role taken by the special educator in co-teaching (Carty & Marie Farrell, 2018).
3. **Station teaching** is a format in which pupils are divided into small groups that rotate between stations, with the mathematics teacher and the special educator leading two stations and no leaders at the remaining stations (Friend et al., 2010). In this type of co-teaching, both educators have an instructional role, which creates a more equal sharing of tasks and responsibilities. Moorehead and Grillo (2013) also suggest that this is an appropriate form of co-teaching for educators who are new to co-teaching, as the roles are relatively independent, and each educator develops a plan for managing their group. The station teaching form reduces the number of pupils per teacher, allowing for more effective differentiation and individualisation. At the same time, this form can lead to behavioural problems for pupils and noise in the classroom, so it is essential to plan activities thoughtfully (Moorehead & Grillo, 2013).
4. **Parallel teaching** is a form in which a mathematics teacher and a special educator teach the same content, differing only in the differentiation of the explanation and the work instructions (Friend et al., 2010). This form of co-teaching also reduces the number of pupils per teacher, but because it is parallel teaching, it also increases the noise levels. It is therefore very important that the mathematics teacher and the special educator time the activities well and adjust them so that they are not disruptive for the other half of the pupils. Parallel teaching in mathematics can be most successful in the consolidation phase, where the

two teachers can ensure that the correctness of the mathematical problems is monitored on an ongoing basis.

5. **Alternative teaching** is a form of co-teaching in which pupils with learning difficulties are grouped together in a small group within the classroom, under the guidance of a teacher and the supervision of a special educator. This allows pupils with learning difficulties to be present in the maths lesson, while receiving more intensive and individualised work instruction from the special educator. The teacher plays a dominant role, and the special educator takes on the role of not only providing on-going support, but also of providing alternative teaching, where his task is to guide the group in such a way that these pupils achieve the learning objectives within the given learning content. This form of collaboration requires close cooperation and a joint search for teaching methods, adaptations and aids that will help the group of pupils with learning difficulties under the guidance of the special educator (Carty & Marie Farrell, 2018).

6. **Teaming** is the form that requires the most planning and cooperation between educators during implementation. As Carty and Marie Farrell (2018) state, teaming depicts a form of teaching in which pupils are presented with an exchange of opinions, viewpoints or dialogue between the teacher and the special educator. In this process, the two professionals may display conflicting opinions or question each other to establish a dialogue and their own reflections (Sileo, 2011).

The Figure 1 shows a schematic representation of the forms described, showing the position of the individual pedagogues and pupils.

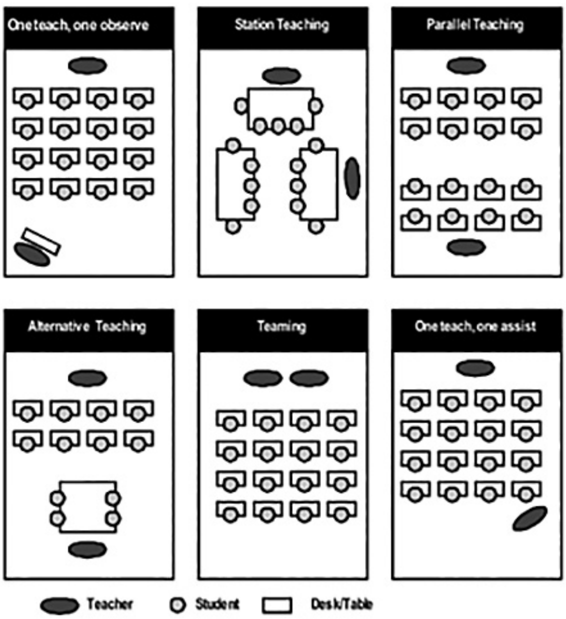

Figure 1. Co-teaching forms (Friend & Bursuk, 2009).

Buli-Holmberg and Jeyaprathaban (2016) note that each form of providing additional support has its advantages and disadvantages. Therefore, it is crucial that the mathematics teacher and the special educator combine their skills and decide on the form of APS implementation in a timely manner, according to the needs of the pupil, and provide the most effective support and assistance.

3. Empirical part

3.1. Purpose of this study

As previously indicated, the implementation of APS in mathematics is a complex process where the mathematics teacher and the special educator need to work together to provide successful support. Since in practice special educators are often faced with the challenges of consolidating and interpreting mathematical content, close collaboration with the mathematics teacher is essential for the development of the didactic-mathematical competences that the special educator needs to provide appropriate support in mathematical content. In the case of co-teaching, the collaboration is even more important and at the same time allows for the integration of competences. We have not found any studies investigating collaboration between mathematics teachers and special educators in Slovenia, so our research focused on the importance of different forms of collaboration and the challenges they face.

The aim of our research was to investigate, in what form the selected schools implement the support, in what way, in what form and how often the special educator and the mathematics teacher collaborate, in which cases, according to the professionals, individual or inclusive implementation is more appropriate, and what are the advantages and disadvantages of APS inside and outside the classroom and to find out how the profile of the professionals in the team providing each form of APS is determined in the selected school.

We are aiming to answer the following research questions:

- How do the selected schools determine the profile of the educator to deliver each form of APS?
- What is the form of the APS in the selected schools?
- In which cases do professionals consider it more appropriate to implement APS within or outside the classroom?
- How, in what form and how often are the special educator and the mathematics teacher involved in the implementation of the APS?
- What are the advantages and disadvantages of implementing APS within and outside the classroom in the selected sample of schools?

3.2. Methodology

We used a qualitative research approach and descriptive method. The data were analysed by qualitative content document analysis.

3.3. Sample

The survey was carried out in 9 randomly selected Slovenian elementary schools between December 2022 and March 2023. Four of them are from North-Western, four from Central and one from South-Eastern Slovenia. The schools are located in urban or suburban areas. In each school, we interviewed the special educator who implements the APS at mathematics and the mathematics teacher who teaches these pupils. In one school, the special educator refused to participate, so we only interviewed the mathematics teacher. In total, we conducted 8 interviews with special educators, including 7 special rehabilitation educators and 1 inclusive educator, and 9 interviews with mathematics teachers. All special educators were female, and among the mathematics teachers we interviewed one male and ensured anonymity for all interviewees.

3.4. Instruments

For the purpose of the research, we designed and conducted semi-structured interview with special educators and mathematics teachers. The interview consisted of 9 open-ended questions.

The interview included questions of the organisation of the APS in the school, to determine the profile of the APS providers and the frequency and modalities of collaboration between the mathematics teacher and the special educator. The questions covered types of providing APS, the forms of collaboration, and the types of information exchanged between the mathematics teacher and the special educator during the collaboration. The last set of questions concerned the interviewees' perceptions of the benefits and challenges of co-teaching.

The instrumentation was developed objectively and reviewed and validated by three experts in the field of mathematics didactics and one in the field of special education. Their expertise contributed to ensuring the quality and objectivity of the semi-structured interview.

3.5. Data collection

Requests for interviews were sent by email. They were sent to 18 schools, of which 9 responded. The interviews were conducted face-to-face in the selected schools. The answers were recorded on a computer and, with prior consent, audio-recorded and later, where necessary, the answers were updated. We conducted a full interview with the APS providers, lasting about 45 minutes each, and a shorter interview of 20 minutes with the mathematics teachers, asking them about their views on working with special educators and the advantages and disadvantages of different forms of APS.

The interview data were broken down into the following segments: *1) The way in which special educators are assigned to individual pupils, 2) Professional profiles of special educators, 3) Forms of APS and frequency of APS, 4) Criteria* *for determining the form of APS, 5) Frequency and forms of collaboration in the case of providing APS inside or outside the classroom,* and *6) Advantages and disadvantages of providing APS inside or outside the classroom*. Following these segments, we will present the findings of the study below.

4. Results

1) The way in which special educators are assigned to individual pupils

The results show that in all but one school, APS providers are assigned pupils in need of APS at the beginning of the school year. These pupils are provided with support in all subject areas in which they need it. Individual pupils are paired with special educators from the beginning to the end of their schooling, except in one school where a second special educator is assigned to the pupil at the transition from the primary to the secondary school level.

In one school, APS providers are allocated hours according to their self-assessed areas of strength. This means that each special educator offers APS only in certain subjects, and thus specialises in that subject area. In addition, at the school of our sample, special educators are divided into classrooms and consequently provide mathematics support to all those who are allocated support in their chosen classroom. They follow these pupils until the end of the school year.

2) Professional profiles of special educators

In most schools, overcoming deficits, barriers or disabilities is mainly done by special and rehabilitation pedagogue. In one school, it is mainly carried out by an inclusive pedagogue who takes over the whole support at secondary school level, except in exceptional circumstances such as accompanying a pupil with an autistic disorder.

In addition, social pedagogues (6 schools) and psychologists (3 schools) also provide support to overcome deficits, barriers, and disabilities in mathematics at secondary school level.

The APS is implemented as a learning support by mathematics teachers in the interviewed schools. In three schools, due to the overload of mathematics teachers, teaching support is also provided by physics teachers, an inclusive pedagogue, a geography teacher, and a librarian.

3) Forms and frequency of APSs

Five schools mostly provide APS outside the classroom, two schools provide it inside the classroom, and two schools have a comparable proportion of both ways of providing APS. In the schools where APS is implemented inside the classroom, *one teaches, one assists* is the predominant form of APS, and in one school *alternative teaching and parallel teaching* are also implemented 2–3 times a year.

Who determines whether APS is implemented inside or outside the classroom varies from school to school. In schools where APS is predominantly carried out of the classroom, it is decided jointly by the special educator and the mathematics teacher whether it would be better to provide support inside or outside the classroom in a particular situation.

In two schools, where both forms are implemented in comparable proportions, the form of implementation is suggested by the mathematics teacher, and in one school the form of implementation is suggested by the APS provider. In one of the schools surveyed, for one pupil, all support is provided inside the classroom, as this was the wish of the parents.

In schools where the majority of support is provided outside the classroom, it is most often the decision of the special educator, and in one case it is the agreement of both the maths teacher and the special educator.

In schools where the form of implementation is not decided by consensus, some mathematics teachers and special educators highlight their frustration at not deciding together.

4) Criteria for determining the form of implementation of the APS

In two of the five schools where APS in mathematics is carried out outside the classroom, it is stressed that APS outside the classroom is not carried out during the explanation of new mathematical content. At that time, the pupil remains in the classroom in order to be present at all times during the mathematics teacher's explanation. In the other two schools, in this case, the explanation of the new mathematical content is carried out by a special educator outside the classroom.

Special educators who also carry out APS with pupils during the teacher's explanation of the new learning content, in this case explain the new learning content to the pupil themselves. They state that they feel sufficiently competent to deliver the new mathematical content because of their long experience.

One of the two schools where APS is implemented inside and outside of the classroom in comparable proportions states that they choose to implement it in the classroom if it involves explaining new learning content, while in the other school this is not a criterion. They indicate that they decide on implementation based on the developmental needs of the pupil, possible emotional distress that is more easily addressed outside the classroom, or social interaction difficulties where it makes more sense for the special educator to work with the pupil inside the classroom.

Interviewees in schools where the predominant APS implementation is inside the classroom (2 schools) indicate that they opt for out of the classroom implementation in cases where the teacher does not provide an explanation of the new learning content. If this criterion is met, they opt for out-of-classroom delivery of APS during the consolidation period before the assessment, if the pupil is experiencing emotional distress or if he/she is far behind in the content.

5) Frequency and forms of collaboration in the case of providing APS inside or outside the classroom

Communication and frequency of team planning

All interviewees communicate in person, five pairs of special educator and maths teachers also communicate via email. They mostly communicate during a

5-minute break just before the start of the lesson. Those who use email usually exchange information a day before the lesson. In a school that also implements co-teaching in the form of *parallel teaching and alternative teaching*, it is stated that if this type of co-teaching is implemented, the mathematics teacher and the special educator meet 3–4 times before to prepare for the co-teaching.

All interviewees state that the time between 5-minute breaks is the only time available for them to participate, as most of them do not want to meet after school, and that communication depends on the initiative and resourcefulness of the participants, as they do not have a pre-set time for meetings from the school management. Twelve interviewees are quite satisfied with communication as it is, while five interviewees express dissatisfaction with the lack of time for joint planning (three of them special educators and two mathematics teachers). The desire for more planning time is reported by all mathematics teachers and APS providers where APS is predominantly delivered within the classroom, and by two mathematics teachers and two special educators from schools where APS is delivered predominantly inside or in comparable proportions outside the classroom. These mathematics teachers would like to be approached by the special educators at least one day before providing APS so that they can discuss and plan the APS in more depth. The special educators would also like to be able to discuss in more detail aspects of the APS that are related to mathematical content for which they do not feel competent.

In three schools, evaluation is carried out after every lesson, in three schools it is only carried out when there are any specificities, and in the remaining three schools it is very rarely carried out.

Content of planning and evaluation – APS outside the classroom

All schools state that if the APS in maths is conducted outside the classroom, the maths teacher will provide the special educator with the content or tasks to be covered or solved during the pupil's absence from the maths lesson. Three teachers also provide the special educator with their own teaching preparation where in one school they provide it one week in advance and the other two providing it one day before or during a 5-minute break before providing APS.

All mathematics teachers indicate that they are ready to present teaching approaches and types of representations if the special educator so wishes. They indicate that it is mainly special educators with less experience who wish to do so.

Two mathematics teachers express dissatisfaction with the collaboration, as they would have liked the special educators to have been informed at least one day before the lesson about what they were going to do and to have had a more detailed discussion about the mathematical didactic aspects or the content of the implementation. Three special educators also state that they would have liked to have met with the teachers at least once a week to discuss mathematical content and appropriate didactic-mathematical approaches.

The implementation of the evaluation after the APS varies widely across the schools interviewed. The schools that do carry out evaluation (6 schools) indicate that it includes information on the pupil's performance in solving the tasks and his/her understanding of the content.

Content of planning and evaluation – APS inside the classroom

In the four schools where APS is implemented at least in a comparable proportion inside the classroom as outside the classroom, three mathematics teachers send their preparation to the special educator one day in advance via email, and in one school the special educator is briefed on the content of the lesson during the break before the lesson. In the break before the co-teaching starts, the interviewed mathematics teachers and special educators also discuss which pupils they expect to have difficulties in the lesson and which concept, or procedure might be problematic. A special educator who learns about the content of a lesson just before the lesson starts points out that he/she cannot prepare adequately for the lesson. Because he/she does not know the content of the lesson in advance, he/she cannot bring to the classroom the materials that pupils with learning difficulties in mathematics might need. In all schools, the evaluation of the co-teaching lesson is carried out by the educators immediately after the lesson, where they exchange information on the progress of individual pupils and their difficulties related to the content of the lesson.

6) The advantages and disadvantages of implementing APS inside and outside of the classroom.

Advantages and disadvantages of implementing APS outside of the classroom as identified by mathematics teachers

The advantages of providing APS outside of the classroom are identified by mathematics teachers as the fact that the special educator can do more tasks in an individual situation with the pupil, and that the special educator has a better insight into the nature or origin of the pupils' mathematics difficulties. They also point out that the work outside the classroom is more adapted to the individual's pace of learning, both the pupil and the special educator are more active, and the atmosphere is more relaxed. They also state the advantage that the special educator can explain the content in a way that is adapted to the pupil's needs.

Mathematics teachers point to the fact that in providing APS outside of the classroom the mathematic content is explained by a special educator who does not have mathematical didactical knowledge as a disadvantage. At the same time, they state that, due to the absence of the pupil in the mathematics lesson, teachers do not have an overview of both the pupil's progress and the content presented to the pupil during the APS.

Advantages and disadvantages of implementing APS outside of the classroom as identified by special educators

Special educators see the advantages of providing APS outside of the classroom as being able to focus more on the pupil, motivate him more easily, have access to more didactic tools, adjust the pace of the work more easily, and adapt the work to the pupil's prior knowledge, emotional state, strengths and weaknesses.

As disadvantages they state the lack of interaction between pupils and their peers, the fact that individual work outside the classroom is more tiring for the pupil, and the fact that some pupils have poorly organized notebooks and homework due to absenteeism.

Advantages and disadvantages of implementing APS inside the classroom as identified by mathematics teachers

As stated by mathematics teachers the advantages of implementing APS inside of the classroom are that it allows pupils to participate more actively in the primary learning environment and to follow the flow of the lesson more easily. They point out that the special educator in the classroom can get a better insight into the pupil's functioning and interactions with the teacher and classmates. As an advantage, they state the fact that the special educator can listen to the explanation of the new learning content while being present in the classroom, which is particularly useful for those with less experience. Another advantage is that the special educator can distribute the help among all the pupils who need it at any given time.

Mathematics teachers in schools where APS is predominantly implemented inside of the classroom state only one disadvantage that is lack of time for joint planning, while in schools where APS is less frequently implemented inside the classroom, teachers identify significantly more disadvantages. They point out that the presence of a special educator in the classroom is distracting, both because of the loudness of the special educator's speech and because of personal preferences. They also feel that the presence of a special educator in the classroom in the form of co-teaching *one teaches, one assists* makes the pupil too exposed.

Advantages and disadvantages of implementing APS inside the classroom as identified by special educators

Special educators in schools where APS is predominantly implemented inside of the classroom state that the social inclusion of pupils and the fact that both pupils and special educators are present in the classroom during the explanation of the new learning content as an important advantage. As a result, special educators have a better overview of the content, can better plan aids and adaptations, and determine the scope of support, which the special educator can then use when providing additional support or explanation to the pupils outside the classroom. They also state the advantage that pupils are less stressed than when they leave the classroom with the special educator. They point out that they can distribute the help more meaningfully, fairly, and evenly among all the pupils who need help. They stress that they can provide help to pupils in the classroom as soon as they encounter a problem, rather than at the next APS lesson outside the classroom. In addition, they see an advantage that in co-teaching the mathematics teacher can more easily differentiate and individualise when a special educator is present in the classroom. They point out that pupils are more motivated to work when a special educator is present in the classroom and that it is easier to teach other skills, such as active listening, with the special educator in the classroom. The more often special educators are present in the classroom, the easier and faster it is for them to identify problems in pupils. They have also observed that when APS was introduced in the classroom, pupils with learning difficulties performed better in the national mathematics assessment, as they were closer to the average score. The only disadvantage is the lack of time for joint planning.

In schools where providing APS inside of the classroom is less frequent, the advantages include less exposure of pupils when they leave the classroom and more effective handling of behavioural problems during mathematics lessons. Also, the presence of the special educator in the classroom allows him/her to get to know the teacher's teaching style and mathematical content and didactic approaches. Some special educators do not recognise any advantages of implementing APS inside the classroom.

Special educators who provide APS inside of the classroom less often or not at all state that the disadvantages of providing APS inside the classroom as the fact that the special educator can be distracting by speaking in the classroom, that special educator has a less active role than in an individual situation, that the special educator can pay less attention to the pupil than in out-of-classroom work, and that this is therefore less effective. They point out that pupils are overexposed in such a setting, which in their opinion has a negative impact on their self-esteem.

5. Discussion

Educating pupils with learning difficulties in mathematics is a complex process, and it is difficult to expect that one professional has sufficient knowledge to meet the pupil's needs independently and successfully (Pulec Lah & Košir, 2015). Like Pulec Lah and Košir (2015) we believe that the educational process of pupils with learning difficulties in mathematics should therefore involve professionals with different skills and competences. Therefore, the cooperation of the special educator and the mathematics teacher is of utmost importance for the successful achievement of the expected (appropriate) level of knowledge, inclusion in the social environment and optimal holistic development of the pupil with learning difficulties in mathematics. Our aim is to present and critically reflect on the different practices implemented in different schools in Slovenia and to highlight those that stand out and represent innovative approaches to the implementation of the APS in our opinion.

To specialise the special educator for giving APS only in mathematics (as demonstrated in one school from our sample) is considered appropriate from a mathematical point of view, but the question is how this affects a pupil who receives help from several different professionals. Kobolt and Rapuš-Pavel (2009) find that opinions differ on whether it is better for a pupil to be assisted by one or more than one person. Some believe that one person could provide a pupil with a higher level of structure and security, while others argue the opposite. Such a deployment of special educators is only possible if there are enough special educators in the school, but at the same time requires a more complex organisation. Although in one of the schools interviewed, special educators distribute the provision of subject-specific support according to their areas of strength, we are aware that having an opinion on one's own competences is not a guarantee of adequate expertise. The subject-based distribution allows special educators to follow a particular pupil throughout his/her time in elementary school in mathematics, thus keeping up-to-date and more holistic monitoring of his/her development, knowledge and progress. Special educator is assigned to a specific classroom where he/she provides support to all pupils. In this way, he/she has more time in each classroom and gets to know the teacher and his/her teaching style better. As a result, the same mathematics teacher and the same special educator accompany the pupil throughout his/her schooling at the secondary school level, thus creating a team with the pupil, where they work together more effectively to achieve their goals.

We are aware that both the special educator and the maths teacher are qualified in the field for which they are being trained. The mathematics teacher is equipped with mathematical-didactic knowledge, has a detailed knowledge of the learning objectives and minimum standards, and uses mathematical language appropriately, while the special educator is familiar with the nature of special needs, strategies to overcome deficits arising from the nature of learning difficulties, compensatory strategies, adaptations and aids. We would therefore like to stress that it is even more important or necessary for the special educator and the mathematics teacher to work closely together, since only by combining their competences can they provide the pupil with effective support in overcoming deficits and adequate help with mathematical content.

If the APS is implemented outside of the classroom, we are faced with the problem that no specialist has both the special-education and mathematics-didactic knowledge that are indispensable for holistic help and support for pupils with learning difficulties. The study finds that some special educators also feel competent to deliver new mathematical content. This is particularly noticeable for special educators who have a long teaching practice and who state that they are familiar with the content of mathematics and the teaching approaches of the teacher they work with. On the contrary, special educators with less experience, consider their knowledge of mathematics content as their weak area. From both the perspective of mathematics teachers and the self-assessed competence of special educators, work experience helps to increase confidence in the independent delivery of mathematical content.

We consider it reasonable that in the vast majority of schools, providing APS in the form of learning support is delivered by mathematics teachers. However, it should also be pointed out that they receive very little exposure to strategies for working with pupils with learning difficulties as part of their studies, and therefore face challenges in the area of special-education strategies. It is considered inappropriate that APS is also provided by the librarian and the geography teacher as they practise in one of the interviewed schools. We are aware that schools' problems with lack of teachers and special educators and, as a consequence, APS in mathematics is also provided by other, less appropriate profiles. From a professional point of view, we consider these practices to be inappropriate, as these profiles do not have the mathematical and didactic knowledge that are necessary for the provision of learning support in mathematics.

As it is very important for pupils, especially those who need extra help, that all professionals use the same mathematical language when teaching and providing extra help, it is essential that a special educator is present during the maths teacher's explanation. The importance of consistent language is also noted by Gersten et al. (2009), who found from research that using appropriate mathematical vocabulary when providing support to pupils enables faster progress. We suggest that the special educator is also present in lessons where the teacher is explaining new content. As the special educator takes a less active role during the teacher's explanation, the opportunity is then offered to monitor the explanation, with an emphasis on monitoring vocabulary, representations of concepts and procedures, the chaining of tasks and objectives, essential questions to check understanding, and the use of appropriate procedures and their justification. We suggest that the special educator keeps the above highlights in written form as a tool that can help him/her and other special educators to overcome pupils' deficits in mathematics content. It is important that he/she also writes down the learning objectives and the name of the teacher to whom he/she listened to the explanation, because teachers are autonomous in the way they deliver the content, taking into account the curriculum, thus unifying his/her own way of explaining with the teacher's and ensuring consistency for the pupil in terms of the mathematical content covered. When a pupil receives a disjointed professional and special didactic support, the results are significantly worse, despite the efforts of both individual educators, than when the mathematics teacher and the special educator work closely together in a pupil-centred way.

Whether the APS is provided inside or outside the classroom, the flow of information between the two providers is important. In line with the findings of the evaluation study (Vršnik Perše et al., 2016), in the selected schools, the information flow between mathematics teachers and APS providers varies and depends on the self-initiative of both professionals. There are no specific rules on when and what information should be provided by the teacher to the special educator prior to the implementation of the APS outside the classroom, so practices vary from school to school. We do not see a solution in setting a deadline for information provision, as the needs of the professionals are very different, but it seems appropriate that the management of the school gives opportunity the APS providers to participate on a weekly basis, as suggested by Opara (2015). This could avoid the lack of planning time, which according to the results of the study mostly takes place only during the five-minute breaks before the APS.

Like many authors (Austin, 2001; Friend et al. 2010; Fennick & Liddy, 2001; Sileo & van Garderen, 2010) researching co-teaching, we find that both mathematics teachers and special educators point to the lack of time to plan and prepare together for co-teaching as a major barrier. Also in the selected schools, collaboration prior to the implementation of the APS, inside or outside the classroom, takes place mainly on an ongoing basis, during the 5-minute breaks before the lesson. We would like to point out that some special educators and mathematics teachers consider 5-minute breaks as a sufficient amount of time for joint planning, which in our opinion is significantly insufficient for planning a quality implementation of any form of APS. The implementation of APS inside the classroom is more demanding from an organisational point of view than the implementation outside the classroom, as it requires continuous collaboration and joint preparation of lessons, which is time-consuming. As a result, in the selected schools, co-teaching is mainly implemented in the form of *one teaches, one assists* (Friend et al., 2010), which is less demanding from a planning perspective. However, even a good quality implementation of this form requires up-front collaboration, which is not sufficient for a 5-minute break before the lesson starts. One of the special educators states the fact that she does not receive any preparation in advance from the mathematics teacher as a problem. Thus, she is not able to prepare well for the APS inside the classroom, as she is not able to provide the appropriate aids for the pupil or to plan adjustments to the provision of assistance. In this way, the *one teaches, one assists* format is not as effective as it could be. Professionals in schools where other forms of co-teaching, such as *parallel teaching* and *alternative teaching*, are occasionally implemented, state that they need to meet several times and well in advance to plan this type of co-teaching. Team preparation for this form of co-teaching therefore requires more time than form *one teach, one assist*. Scruggs et al. (2007) and Solis et al. (2012) cite the systematic organisation of time for joint planning of co-teaching as an organisational prerequisite. Fennick and Liddy (2001) point out that regular and planned meetings of professionals are necessary for the optimal success of co-teaching. As suggested by Opara (2015), some special educators and mathematics teachers also point out that collaborative team meetings should be facilitated by the school at least on a weekly basis. Some special educators also express a desire to do so, mainly due to their lack of knowledge in the area of mathematical content, in which they would like to receive further training. Research (Hill et al., 2005) shows that mathematical content knowledge of special educators has a significant impact on pupils' learning achievements. Therefore, we summarise that special educators need to be empowered in this area as well. As a possible solution, we see pre-planned weekly meetings of special educator-teacher mathematics teams, which would be aimed at information flow and knowledge sharing in the case of individual implementation, or joint planning of co-teaching in the case of providing APS inside of the classroom. This would help improve the quality of the educational process, as co-teaching has positive effects on the professional development of the professionals involved. Special educators would thus gain knowledge in the subject content area, while teachers would gain knowledge in the area of curriculum adaptation (Austin, 2001; Ducman, 2012). We also see providing APS in form of co-teaching as a form of teaching that allows bridging the gaps in the competences of special educator and mathematics teacher.

Sileo (2011), Trent et al. (2003) and Moorehead and Grillo (2013) also highlight the importance of a clear and pre-agreed division of roles between special educator and mathematics teacher inside the classroom, as otherwise it is likely that the special educator will only play the role of observer and occasional support and assistance, which is the most common practice confirmed in our study. Scantlebury et al. (2007, in Moorehead and Grillo, 2013) emphasised the need to systematically develop in-depth collaborative skills focused on teaching and pupil learning. This is more difficult to develop in mathematics and science subjects if there are gaps between the willingness and knowledge of both professionals. In our sample of elementary schools, there is also a lack of clarity in the division of roles and responsibilities in some cases. We suggest that at the start of the collaboration, the mathematics teacher and the special educator discuss in detail the roles, objectives, and responsibilities of the collaboration, while planning time for evaluation of the implementation of the co-teaching and adapting it accordingly to the needs of the individual.

Despite the success of the collaboration between the special educator and the mathematics teacher and the many advantages of co-teaching, questions arise as to which forms of APS in mathematics make sense. In the case of providing APS outside of the classroom, we return to the question of the competence of special educators who are not equipped with the necessary mathematical-didactical knowledge to teach mathematics. In the case of providing APS inside of the classroom, the dilemma arises as to whether the main factor in the implementation of the various forms of APS is really the time devoted to planning. It is also necessary to question the rationale for implementing a particular form of co-teaching given the specificity of the mathematics content and to define the form in terms of co-teaching.

Undoubtedly, the form of co-teaching used in mathematics teaching depends on the mathematical content and the method judged most appropriate to address that content. Table 1 shows some of the possible applications of co-teaching forms in selected teaching methods.

Table 1. Examples of co-teaching forms implementations in mathematics classroom.

Teaching method	Form of co-teaching	Examples
Consolidation	One teaches, one assists	Special educator provides help in solving geometry problems by using geometry models.
Problem solving and consolidation	Station teaching and alternative teaching	One group of pupils is provided additional explanation and uses models for solving particular problems in geometry (e.g. volume of pyramids).
Explanation of mathematical process	Teaming	Showing different strategies for solving word problems and discussing them in team by focusing on misconceptions pupils might encounter.
Explanation of new mathematical content	One teaches, one observes	Listening to the explanation of mathematics teacher of long division and using the same adapted explanation in additional support outside the classroom.

In conclusion, it should be stressed that there is always a need to give careful consideration to which form of delivery (in or out of the classroom) is chosen, taking into account the needs of the pupils. Recognising the diversity of individual situations, the specificity of the needs, difficulties and deficits of the pupils, the personal characteristics and training of the different professionals, the goodwill of the school and the availability of planning time, such decisions on the form of implementation of the APS need to be well thought out in each team.

Summary

Providing additional support in mathematics is a complex process, specific to the characteristics of the individual pupil. Recognising that there is no rule for the optimal way to provide support (within the constraints of the law), we will summarise the findings and examples of what we consider to be good practice that we have observed in our research, in order to encourage reflection by all those involved in the process of providing support to pupils with learning difficulties.

Since didactic-mathematical knowledge is key to helping pupils with learning difficulties in mathematics, it seems reasonable that, in addition to mathematics teachers, support should be provided by special educators who feel competent to deal with mathematics. Although competence is strongly linked to one's own subjective beliefs, these may be the key to motivation and the desire to improve in the area of content knowledge in mathematics.

As the APS is linked to the specific problems of the individual pupil, the forms of APS differ from one another. The central conceptual division that divides APS into learning support and overcoming deficits also separates the professionals who provide each form of support. For example, learning support is mainly delivered by mathematics teachers, while overcoming deficits, barriers and disabilities is delivered by special educators. In terms of delivery, APS is divided into APS delivered outside and within the classroom. Deciding which form of APS to implement is not so simple. It is necessary to put the pupil at front, to adapt to his specificities and to find ways to implement APS in a form that will benefit the pupil the most. Each of these forms requires careful thought and collaboration between the mathematics teacher and the special educator. In the case of providing APS outside the classroom, there is the question of the mathematical didactical knowledge of the special educator, who bridges deficits, barriers or disruptions in the context of the mathematical content, but in the providing APS inside the classroom, a well-implemented co-teaching lesson requires sufficient planning and evaluation time to achieve the desired results. School management has a very important role to play here and should plan the planning time in advance for the special educator and the mathematics teacher if they wish to implement co-teaching.

We would like to emphasise that we cannot determine whether it is better to implement APS inside or outside the classroom in mathematics, but we see co-teaching as a form of teaching where professionals, with enough planning, can combine their competences and help the learner in a holistic way.

Limitations of the study

We are aware of the limitations of the study, as it was carried out on a relatively small sample. It would be useful to carry out interviews in more schools to gain a better insight into the diversity of APS practices. It would also be useful to carry out observations of different forms of APS implementation, which would allow for more objective data on the strengths and challenges of different forms of APS implementation.

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Pomen sodelovanja učitelja matematike in specialnega pedagoga pri nudenju pomoči učencem z nižjimi dosežki pri matematiki

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Povzetek. Število učencev, ki imajo prilagojeno izvajanje pouka oz. so deležni dodatne strokovne pomoči pri matematiki vsako leto narašča. Zagotavljanje kakovostnega izvajanja pomoči za učence, ki imajo nižje dosežke pri matematiki, je na področju izobraževanja matematike vedno bolj aktualno, med drugim se pojavlja vprašanje, kako zagotoviti kompetentne strokovnjake za izvajanje dodatne strokovne pomoči. Matematika na predmetni stopnji obsega takoj za slovenščino največ ur, je zahtevna, abstraktna ter vedno manj vezana na materialno realnost. Na tej stopnji pri matematiki ni več mogoče strogo ločiti premagovanja primanjkljajev v znanju učencev pri posameznih matematičnih vsebinah od učne pomoči, zato je nujno učinkovito sodelovanje različnih strokovnjakov, ki v učni proces doprinesejo kompetence in znanja s svojega strokovnega področja. Trenutno v slovenskih osnovnih šolah prevladuje individualna izvedba dodatne strokovne pomoči pri matematiki izven razreda, vendar strokovnjaki poudarjajo, da bi bilo tako z vidika doseganja optimalne učne uspešnosti kot tudi socialne vključenosti potrebno pomoč učencu v večji meri nuditi znotraj razreda (Pulec Lah in Košir, 2015). Nekatere slovenske šole pričenjajo s tovrstno prakso, ki pa zahteva veliko timskega sodelovanja predmetnega učitelja matematike in izvajalca dodatne strokovne pomoči ter učinkovito organizacijo. Cilj naše raziskave je bil raziskati, v kakšni obliki na izbrani šoli izvajajo dodatno strokovno pomoč, na kakšen način, v kakšni obliki in kako pogosto pri tem sodelujeta specialni pedagog ter predmetni učitelj matematike, v katerih primerih je po mnenju izvajalcev bolj ustrezna individualna ali inkluzivna izvedba ter katere so njune prednosti in pomanjkljivosti. Skladno s Pravilnikom o izobrazbi učiteljev in strokovnih delavcev v izobraževalnem programu osnovne šole (2022) dodatno strokovno pomoč izvajajo različni profili strokovnih delavcev, zato je bil naš namen ugotoviti, na kakšen način določajo profil strokovnega delavca za izvajanje posamezne oblike dodatne strokovne pomoči na izbrani šoli. S poglobljeno analizo sistematičnega opazovanja individualnega in inkluzivnega poučevanja matematike in intervjujev s predmetnimi učitelji matematike ter specialnimi pedagogi smo ugotovili, da je učinkovito sodelovanje med predmetnim učiteljem matematike in specialnim pedagogom ključno za doseganje učnih ciljev pri poučevanju matematike. Kakovostna izvedba poučevanja učencev z nižjimi dosežki pri matematiki zahteva kakovostno timsko sodelovanje vseh udeleženih. V zaključkih predlagamo, da bi učitelji matematike in specialni pedagogi v sodelovanju načrtovali učne ure za matematiko v primeru inkluzivne izvedbe, predvsem pri zahtevnejših vsebinah. V kolikor specialni pedagog nudi učencu individualno strokovno pomoč predlagamo, da je ob inkluzivnem izvajanju učnih ur specialni pedagog prisoten pri razlagi učitelja matematike in si sproti v, za ta namen pripravljen, obrazec vpisuje ključno matematično besedišče, reprezentacije pojmov in postopkov, veriženje nalog ter ostale pomembne napotke, ki jih lahko uporabi pri delu z učencem izven razreda z namenom, da zagotovi konsistentnost obravnavanja pojma. To specialnemu pedagogu predstavlja predpogoj za vzpostavitev dialoga z učenci, ki pa ga nadalje razvije skladno s specialno pedagoško stroko. Kadar je učenec deležen nepovezanega strokovnega in specialno didaktičnega načina nudenja pomoči, so rezultati kljub prizadevanjem pedagoških delavcev bistveno slabši, kot če učitelj matematike in specialni pedagog tesno sodelujeta osredinjeno na učenca.

Ključne besede: specialni pedagog, predmetni učitelj matematike, dodatna strokovna pomoč, sodelovanje strokovnjakov, inkluzija

Superficial Strategies in Solving Compare-Combine Word Problems

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Abstract. Compare and combine word problems are used in elementary mathematics as a part of standard teaching practice. Their integration enables creation of various types of word problems with different structure and level of difficulty. One of the main obstacles in solving word problems is the use of superficial strategies in which students directly translate words they have recognized as entities and relations into mathematical operations and expressions, without understanding the situational model of the problem. The aim of this paper is to investigate the use of these strategies in solving integrated compare-combine word problems. For this purpose, we posed word problems with varying correspondences between entity keywords and relations given in the text of the problems. One hundred and thirty-four students participated in the study by solving paper and pencil test. Forty-four students were in 2nd grade (7,5 to 8,5 years old students), 48 in 4th grade (9,5 to 10,5 years old students), and 42 in 6th grade (11,5 to 12,5 years old students). Results showed that students did not have different achievement on word problems with different correspondence between entity keywords and relations. The superficial approach they used most often was in identifying relational terms (mathematical operations). As expected, there were differences in achievement and in nature of mistakes regarding students' level of education (2nd, 4th or 6th grade).

Keywords: compare word problems, combine word problems, problem solving strategies, word problems

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1. Introduction

Word problems are considered a basis for learning mathematics at the elementary school level. As such, a significant amount of research has been directed towards analyzing different word problem types and identifying the obstacles students face when solving them. Classification of word problems with one mathematical operation on change, combine and compare problems was used as a starting point in many studies. Those studies imply that even though word problems can be solved with the same mathematical operation, they actually belong to different semantical types, trigger different ways of representing and solving, and reveal different types of students' mistakes and misconceptions (Fuson, 1992). More recently, researchers have recognized the importance of problems with higher complexity that integrate change, combine, and compare problems (Nesher et al., 2003). These integrated word problems can be particularly challenging as each sub-problem brings in difficulties that can be attributed to its category, and the integration itself produces problems with varying structures and complexities. According to studies that confirm the relevance of correspondence between the order of information presented in the text of the word problem and the steps in its solving (Daroczy et al., 2015; Vicente et al., 2007), one of the difficulties in integration could also be in the correspondence between the order of entities (in the text of the problem) and order of description (the way in which relations are presented in the text). In this paper, we investigate the importance of this correspondence to understand the use of superficial strategies in students' word problem solving.

2. Students' difficulties in solving compare word problems

During the 1980s researchers singled out three groups of word problems: combine, change, and compare problems (e.g., Riley & Greeno, 1988). The values that are unknown in each problem determine its structure and its level of difficulty. Riley and Greeno (1988) identified combine word problems with an unknown total amount (e.g. Joe has 3 marbles. Tom has 5 marbles. How many marbles do they have altogether? (p. 53)) or subset (Joe and Tom have 8 marbles altogether. Joe has 3 marbles. How many marbles does Tom have? (p. 53)) and compare word problems with an unknown difference set (Joe has 8 marbles. Tom has 3 marbles. How many marbles does Tom have less than Joe? (p. 54)), compared set (Joe has 8 marbles. Tom has 5 marbles less than Joe. How many marbles does Tom have? (p. 54)) or referent set (Joe has 3 marbles. He has 5 marbles less than Tom. How many marbles does Tom have? (p. 54)). Various studies, including those by Stern (1993) and Boonen and Jolles (2015), have shown that combine word problems with an unknown total amount are the least difficult for students, while compare word problems with an unknown referent set are the most challenging. Many studies have attempted to explain why compare word problems with an unknown referent set are difficult to solve. One point of view is that these problems can have different language consistencies. Lewis and Mayer (1987) pointed out that compare word problems can have consistent language (CL) or inconsistent language

(IL) formulation. For example, in CL problem the term "more" in the text of the problem can be successfully solved using addition, while in the problems with IL formulation with the term "more" in the text, addition cannot be used to solve the problem. They found that students make more mistakes in IL problems because they automatically activate the rule "add if the relation is more than and subtract if the relation is less than". This is known as the consistency effect and is confirmed using samples of students from elementary school to college (e.g., Hegarty et al., 1995; Pape, 2003; Stern, 1993). To explain these findings, Stern (1993) and Okamoto (1996) pointed out that students do not understand the symmetrical relationship between relations "more than" and "less than", which is necessary for solving these types of problems.

Boonen and Jolles (2015) conducted a study with second-grade students which did not confirm the consistency effect. The students who were explicitly taught the relations "more" and "less" performed equally well on IL and CL problems.

However, there are studies where verbal instruction on how to solve CL and IL problems did not reduce the consistency effect (Dewolf et al., 2014; de Koning et al., 2017). In one study (Dewolf et al., 2014) students had verbal instruction on the test that informed them that word problems could have different language constructions. In other words, it was explained that the word problem could have one of several types of relational keywords, and that it is important to pay attention to the use of the correct operation. These instructions did not significantly affect the students' achievement. In contrast, in the study conducted by de Koning et al. (2017), verbal instructions were given on how to solve CL and IL problems, with an emphasis on the interpretation of relational keywords to avoid mistakes in the choice of operation. The effect of these instructions was significant on problems that included the keywords "less" and "more", but not on problems that included other keywords such as "higher/lower" or "more expensive/cheaper".

In previous years, research was also directed to the modeling process – the cyclic process of solving word problems that starts from real situation, goes over the use of real models and mathematical models and ends with the mathematical results and real results (Blum & Leiss, 2007). Situational understanding ("understanding situation described in the problem in order to reduce it to its gist") plays a vital role in bridging the gap between language comprehension and mathematical problem solving (Stern & Lehrndorfer, 1992, pp. 261). Linguistic models of word problem solving emphasize the role of language in understanding, with situational understanding seen as a process of "going beyond the text's" (according to Stern and Lehrndorfer, 1992, pp. 261). Students develop an adequate mental representation of the situational model and then translate it into a mathematical model through mathematization (Stern & Lehrndorfer, 1992; Blum & Leiss, 2007). In the modelling process, the mathematical model serves as a foundation for planning and utilizing necessary mathematical operations. Students make mathematical models based on the realistic situation presented in the text of the problem and then they solve the problem in a mathematical context, by performing mathematical operations. In other words, the problem can be formulated in a way that makes semantic relations between the relations in the text more explicit and transparent, and therefore easier for students to solve (Vicente et al., 2007). If the information used to describe the realistic situation follows the order of steps in the solving, students are more likely to succeed (Daroczy et al., 2015). The negative effect of problems formulated this way is that students tend to rely on superficial strategies, such as the “keyword” or “number grabbing” strategy in word problem solving (Briars & Larkin, 1984; Littlefield & Rieser, 1993). Students who transform numbers and keywords directly into arithmetic operations and attempt to “combine” them to find the solution do not construct adequate models for word problem solving (Hegarty et al., 1995). Although one group of problems is formulated in a way that enables students to solve them successfully using this approach, it only promotes the practice of their computational skills and not conceptual understanding and mathematical thinking (Boesen et al., 2014). Our choice to investigate the understanding of compare word problems on problems with complex structures is based on the view that mathematical reasoning could be investigated on the problems that require analysis of the meaning and the structure of a problem, as well as justification of procedures and solving strategies (Stein et al., 2000). Following the authors who use “keywords” for the direct transformation of words “more or less” to mathematical operations, we use “entity keywords” for the direct transformation of entities into numbers without considering the relations between them provided in the text of the problem. For example, in the word problem “Joe has 3 marbles. Tom has 5 marbles. How many marbles do they have altogether?” Joe and Tom are entity key words that students grab and replace them with the numbers 3 and 5.

3. Integrating combine and compare word problems

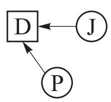
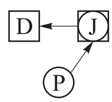
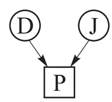
Nesher et al. (2003) investigated the integration of combine and compare word problems, but they used more complex way of integrating, in which the combine problems have unknown subset and compare problems have multiplicative comparisons (Nesher et al., 2003). Their research is conducted with teachers and 15-year-old students and as a result, authors proposed the model of complexity. Two of the variables that Nesher et al. (2003 p. 151) used and that we will use to interpret our results are “number of quantities that are being compared to the number of reference quantities” and “the order of presenting the elementary comparison relations”. The variables will be illustrated later through an example (Table 1, rows Reference structure and Order of description). In our paper, we aimed to examine the integration of compare-combine word problems with a simple structure that can be solved by elementary school students. Therefore, we choose compare-combine word problems with an unknown total number of elements (rather than an unknown subset) and additive comparison relations (rather than multiplicative). As we already stated, the semantic structure of these types of problems can be reflected in referent structure and order of description (Nesher et al., 2003) as well as in different language consistency (Lewis & Mayer, 1987; Hegarty et al., 1995; Pape, 2003; Stern, 1993).

4. Method

The results presented in this paper are part of a larger study investigating students' understanding of relational terminology and their achievement in solving compare-combine word problems. The aim of this paper is to investigate the differences in students' achievement in solving word problems with diverse correspondence between the order of entities and the order of description (relations given in the text of the problems). The use of superficial strategies by students in solving problems would reveal their obstacles in solving problems in which the entity order and order of description do not correspond. To achieve this aim, we defined three types of problems (presented in Table 1):

— P1, which describes the relation between the first (known) entity (David) and the other two (John and Peter);
— P2, which describes the relation between the first (known) entity (David) and the second entity (John), and then between the second (John) and the third (Peter); and
— P3, which describes the relations between the third (unknown) entity (Peter) and the other two entities (David and John).

Table 1. The structure and formulation of compare-combine problems.

	Problem 1 (P1)	**Problem 2 (P2)**	**Problem 3 (P3)**
Text	David has 20 marbles, which is 15 less than Peter and 5 less than John. How many they have altogether?	David has 20 marbles, which is 5 less than John, while John has 10 marbles less than Peter. How many they have altogether?	David has 20 marbles. Peter has 15 more than David, and 10 more than John. How many they have altogether?
Given connections			
Reference structure	1 (compared) to 2 (referent) sets	2 (compared) to 2 (referent) sets	1 (compared) to 2 (referent) sets
Entity order	David, Peter, John	David, John, John, Peter	David, Peter, David, John
Order of description	$D = f(P), D = g(J))$	$D = f(J), J = g(P))$	$P = f(D), P = g(J))$
Language consistency	IL, IL	IL, IL	CL, IL

In P1 and P3 the order of entities (the order of given names David, John, Peter) does not correspond with the relations described in the text. In problem P1 students have to understand that although the order of entities in the problem text is David, John, Peter, the relations described in the text are between David and John, and David and Peter ($D = f(J)$, $D = g(P)$). Similarly, in problem P3, the entity order is David, Peter, David, John, but the relations described in the text are between Peter and David, and Peter and John ($P = f(D)$, $P = g(J)$). On the other hand, in P2 the order of the entities (David, John, John, Peter) corresponds with the relations between the sets ($D = f(J)$, $J = g(P)$). We have formulated all three problems using inconsistent language because problem P3 (which has relations between the third and the second entity) cannot be formulated using exclusively consistent formulations.

Students at different levels of education use various strategies to solve problems. Therefore, we conducted research to investigate the differences in achievement on several levels. We selected 2nd graders because they are familiar with arithmetic strategies for solving word problems and to mathematical texts to some extent, 4th graders because they are more fluent with arithmetic strategies and also have some familiarity with the beginning of algebra, and 6th graders because they are fluent in algebraic strategies. The problems in our study could be solved using arithmetical or algebraic notation and strategies. It is not expected that students fluent with algebraic notation have difficulties in writing and solving system of equations. Hence, we expect older students to be better in solving these problems.

The research sample consists of 134 students from one primary school in Belgrade, which cooperates with the researchers' institution. The students are from two classes of 2nd grade (44 students), two classes of 4th grade (48 students), and two classes of 6th grade (42 students).

Our research questions are:

1) Are there differences and associations in students' achievement in problems with different semantic structures (P1, P2, and P3)?
2) Are there differences in achievement in every type of word problem (P1, P2, and P3) among 2nd, 4th, and 6$^{\text{th}}$ grade students?
3) What are the strategies that students use in solving combine-compare word problems, and what are the most common mistakes they make?

The students were not given a time limit to solve problems P1, P2, and P3. To eliminate the potential influence of task order on their performance, the students were assigned to groups with varying task sequences. For the analysis, we used SPSS and conducted McNemar's exact test, Chi-square test of independence and homogeneity, and phi coefficient to investigate the differences and associations between the achievement in solving word problems P1, P2, and P3. We used 0.05 as level of significance in all tests, and considered association to be moderate if phi is greater than 0.3, and strong if phi is greater than 0.5 (Pallant, 2009). After Chi-square test that compare three variables with significant differences, we performed post hock tests to reveal which pair of variables differ. Additionally, we identified and categorized common mistakes (superficial strategies) made by the students.

5. Results

The students' achievements in solving compare-combine word problems are presented in Table 2, and the results of McNemar's and Chi-square test of independence for investigating the difference and association between problems P1, P2, and P3 are presented in Table 3.

Table 2. Number (percentage) of correct answers to every problem.

Grade	P1	P2	P3
2^{nd} ($n = 44$)	12 (27.3 %)	14 (31.8 %)	16 (36.4 %)
4^{th} ($n = 48$)	24 (50.0 %)	29 (60.4 %)	31 (64.6 %)
6^{th} ($n = 42$)	32 (76.2 %)	33 (78.6 %)	34 (81.0 %)

Table 3. The values of McNemar's test and Chi-square tests.

grade	Tests	P1/P2	P1/P3	P2/P3
2^{nd}	McNemar's	0.687	0.344	0.687
	Chi (44, 1)	20.184, $p = 0.000$ phi = 0.677	10.644, $p = 0.001$ phi = 0.492	21.611, $p = 0.000$ phi = 0.701
4^{th}	McNemar's	0.125	**0.039**	0.754
	Chi (48, 1)	25.176, $p = 0.000$ phi = 0.724	20.493, $p = 0.000$ phi = 0.653	14.977, $p = 0.000$ phi = 0.559
6^{th}	McNemar's	1.000	0.727	1.000
	Chi (42, 1)	6.364, $p = 0.012$ phi = 0.389	8.155, $p = 0.004$ phi = 0.441	1.516, $\boldsymbol{p} = \mathbf{0.218}$

The results show us that there are no statistically significant differences between any pair of problems, as the values of McNemar's tests are greater than 0.05 (Table 3), except for P1 and P3 in 4^{th} grade (McNemars's $p = 0.039$, Table 3) The average success rate for 2^{nd} graders was about 32 %, while the average success rate for 6^{th} graders was about 79 % (Table 2). The 4^{th} graders performed differently on tasks – the highest achievement was on P3 (65 %, Table 2) and the lowest on P1 (50 %, Table 2). Levels of significance of Chi square tests and phi coefficients presented in Table 3 shows us moderate ($p < 0.05$, $0.3 <$ phi < 0.5) to strong ($p < 0.05$, phi > 0.5) associations between every pair of problems except between problems P2 and P3 where association is missing – Chi $(42, 1) = 1.516, p = 0.218$.

The Chi-square test of homogeneity, which examined the differences in achievement among students of different age (Table 4), showed significant differences among all of them for every problem ($p = 0.000$). The post hoc test also revealed significant differences between almost all pairs of grades across all problems, except for P2 and P3, where 4th and 6th graders demonstrated similar achievement ($p > 0.05$).

Table 4. The results of Chi-square test and post hoc tests in comparing success of 2^{nd}, 4^{th}, and 6^{th} graders in solving problems.

		Problem 1	**Problem 2**	**Problem 3**
$2^{nd}/4^{th}/6^{th}$ grade	Chi (134, 2)	20.589	19.551	18.404
	p	0.000	0.000	0.000
2^{nd} /4^{th} grade	Chi (92, 1)	4.978	7.542	7.316
	p	0.026	0.006	0.007
2^{nd} /6^{th} grade	Chi (86, 1)	20.579	18.952	17.554
	p	0.000	0.000	0.000
4^{th} /6^{th} grade	Chi (90, 1)	6.537	3.445	2.992
	p	0.011	**0.063**	**0.084**

To interpret the results, we used types and frequencies of students' incorrect answers (Table 5). The most common mistakes made by students are as follows:

— Incorrect relational term, which occurs when students use the wrong (opposite) mathematical operation. For example, when solving problem P2, "David has 20 marbles, which is 5 less than John. . . " students wrote 20 – 5 for the number of John's marbles.

— Incorrect entity, which occurs when students use the wrong entity in relations. For instance, when solving problem "David has 20 marbles, which is 15 less than Peter and 5 less than John. . . " students made relations between Peter and John instead of David and John.

— Incorrect relational term and entity, which is when students make both mistakes.

— "Keyword" approach, which is when students directly translate words from the problem text into mathematical operations between the numbers in the text. For example, when solving a problem "David has 20 marbles, which is 15 less than Peter and 5 less than John. How many do they have altogether?" students wrote $20 - 15 - 5$ as the answer.

Table 5. Types and frequencies of students' mistakes on P1, P2, and P3, and number of students who made a certain type of error at least once (No).

Grade	**2^{nd}**				**6^{th}**				**4^{th}**			
	P1	P2	P3	No	P1	P2	P3	No	P1	P2	P3	No
Incorrect relational term	4	8	4	**11**	8	10	5	**16**	3	5	5	**11**
Incorrect entity	5	2	2	**8**	1	0	4	**5**	1	1	1	**3**
Incorrect relational term and entity	5	1	4	**9**	1	0	0	**1**	0	0	1	**1**
"Keyword" approach	9	11	9	**12**	1	2	1	**2**	1	0	1	**1**
Total	23	22	19	**40**	11	12	10	**24**	5	6	8	**16**

6. Discussion

Our first research question aimed to explore the differences and associations in students' achievement on compare-combine word problems with diverse correspondence between the order of entities and the order of descriptions. Contrary to our expectations, the results showed that students in 2nd and 4th grade solved problems equally successfully, regardless of this correspondence. There were no statistically significant differences between pairs of problems in 2nd and in 6th grade (Table 3) and the students' average achievement was about 32 % and 79 % in 2nd and 6th grade, respectively (Table 2). These results are to a certain extent opposite to our expectations based on previous research (Daroczy et al., 2015; Vicente et al., 2007) which showed the relevance of correspondence between the order of information and steps in problem solving. We anticipated that students would achieve the highest scores on problem P2, in which the entity keyword and the order of description correspond. However, it seems that language consistency disrupted students' word problem solving. If the problem was formulated in a consistent way, students could solve it by using keyword approach as a superficial strategy (Briars & Larkin, 1984; Littlefield & Rieser, 1993). However, problem P2 was formulated with inconsistent language formulation, so students' use of keyword resulted in an incorrect understanding of the relational term and incorrect choice of mathematical operation. This is in accordance with many studies that investigated inconsistent language word problems (Lewis & Mayer, 1987; Stern, 1993; Okamoto, 1996; Hegarty et al., 1995; Pape, 2003). Students' difficulties with inconsistent formulation of P2 were also confirmed in the analysis of students' incorrect responses. The largest number of incorrect relational term answers were on problem P2 (8 in 2nd, 10 in 4th, and 5 in 6th grade, Table 5), while the smallest number of incorrect answers on P2 was related to an incorrect entity (2, 0 and 1 respectively in 2nd, 4th and 6th grade, Table 5).

In addition, difficulties in solving tasks with no correspondence between entity word and order of description (problems P1 and P3) are reflected in students' mistakes in which they used the wrong entity when solving word problem. The frequencies of these mistakes are provided in Table 5 as "Incorrect entity" and "Incorrect relational term and entity". These incorrect answers were made by seventeen 2nd graders ($8 + 9$ in the "No column", Table 5), six 4th graders ($5 + 1$), and four 6th graders ($3 + 1$). As expected, the mistakes are made on P1 and P3, but they were not frequent enough to make the differences in achievement on problems P1, P2, and P3 in 2nd and 6th grade.

In the 4th grade, there were no significant differences between success in problems P1 (50 %, Table 2) and P2 (60 %, Table 2) and between P2 and P3 (65 %, Table 2), while the difference between P1 and P3 was significant. The number of incorrect answers that students in 4th grade made on P1 and P3 was practically equal (11 on P1, and 10 on P3, Table 5). Therefore, the reason for the significant difference could be found in other factors that define the structure of the problem. The inconsistent language formulation of problem P1 leads to a mathematical model in which two unknown sets are referent, and the other (known) set becomes compared. This is, according to previous research, the most complex compare problem (Hegarty et al., 1995; Pape, 2003; Stern, 1993; Okamoto, 1996). In the third problem (P3) there are also two referent sets that are compared to one (unknown) set, but the relations are presented in the way that only one inconsistent language formulation is used. Considering that the 4th graders had the most incorrect answers that included incorrect relational term (Table 5), we hypothesize that the consistency effect is (at least) one of the reasons for students' higher achievement on problem P3 (with one CL and one IL formulation) than on P1(with two IL formulations), as it was case with studies we mentioned previously in the paper (Boonen & Jolles 2015; Dewolf et al., 2014; de Koning et al., 2017; Pape, 2003; Stern, 1993; Okamoto, 1996).

Moderate to strong associations between P1, P2, and P3 (Table 3) confirm the finding that students are equally successful in solving compare-combine word problems, regardless of the correspondence between entities and the order of description. This result was expected as we used problems with a more complex structure that are more reliable for investigating students' mathematical thinking, which was recommended by Stein et al. (2000). The lowest phi coefficients are between P1 and P2, and P2 and P3 in the 6th grade (phi < 0.5, Table 3), while the association between P2 and P3 is even missing. This implies that the correspondence of entity and order of description is less important for the students in 6th grade than for lower graders (2nd and 4th). An analysis of students' incorrect responses confirms this result. Incorrect response that includes incorrect entity reveals the use of superficial strategies. Students in 6th grade had a low number of these incorrect responses (4 incorrect answers, Table 5). The most frequent obstacle for the students in 6th grade was using incorrect relational terms (12 incorrect answers, Table 5).

Our second research task aimed to investigate the differences in students' achievement across different age. The results of Chi-square homogeneity tests results (Table 4) showed significant differences in performance across all problems. As expected, 6th graders were the most successful, followed by 4th graders and 2nd graders (Table 2). However, there were some exceptions found in the post hoc tests, specifically on problems P2 and P3, where 4th and 6th graders achieved similar scores (Table 4). We suppose that this is due to the lack of a clear gradation of achievement on P1, P2, and P3 in the 6th grade, in which achievement rise from 76.2 % on P1, over 78.6 % on P2, to 81.0 % on P3. This growth is steeper in the 4th grade (from 50.0 % on P1, over 60.4 % on P2 to 64.6 % on P3), hence the achievements of 4th and 6th graders got close enough on problems P2 and P3 to make insignificant difference. Nonetheless, there was still a difference in achievement between 4th and 6th graders on P1, which we attribute to its more complex mathematical structure, involving two unknown referent sets and one known compared set. Only on this problem, with the most complex structure, we have a significant difference in achievement between students of different age.

Finally, the analysis of students' incorrect answers and strategies gives us a closer look at their process of solving compare-combine word problems. Students made the biggest number of mistakes by using the incorrect (opposite) relational term (11 students in 2nd grade, 16 students in 4th grade, and 11 students in 6th grade,

Table 5). In other words, 25 % of 2nd graders, 33 % of 4th graders, and 26 % of 6th graders made relational term mistake. This implies that the consistency effect, that was investigated and confirmed in many studies (Hegarty et al., 1995; Pape, 2003; Stern, 1993), seems to be present at all levels of education, and it is still a significant obstacle for solving these kinds of word problems. It is surprising to note that the 4th graders made more incorrect relational term responses than the 2nd and 6th graders.

Another mistake is using the keyword approach, where students follow words in the problem and transform them into mathematical operations, like in other studies (Briars & Larkin, 1984; Hegarty et al., 1995). We observed that 12 students in 2nd grade made this kind of mistake, which is about one-fourth of 2nd graders, while only a few students in 4th and 6th grade (about 5 % of students, Table 5). This means that the keyword approach is present in 2nd grade, while 4th and 6th graders realize that this strategy will not take them to the correct solution. Littlefield and Rieser (1993) used the term number grabbing for keyword approach. Interestingly, students in our research also used number grabbing – they tend to single out numerical and relational data from the text, without considering entities in the text, nor the meaning of the situational model which researchers (Stern & Lehrndorfer, 1992; Blum & Leiss, 2007) find crucial for word problem solving. For these students, the result of the problem is simply the result of the operation between "grabbed" numbers, which was also the case in other studies (Boesen et al., 2014; Briars & Larkin, 1984; Littlefield & Rieser, 1993).

Surprisingly, none of the students used algebraic strategies for problem solving. This is surprising because the curriculum in our country requires arithmetical and algebraic strategies in problem solving on elementary level (first four grades) and use of mathematical modeling. However, the types of the problems, their extent, complexity and strategies in solving are left for the teachers' choice. In our research, only a few students used algebraic syntax to write down the relations between entities, and then continued with an arithmetic strategy for solving the problem. One 4th grade student used algebraic symbols to represent relations in problem P1, three students in P2 (one 2nd grader, one 4th, and one 6th grader), and one (4th grader) in P3, but none of them managed to set up and solve the equation. We expected that 6th grade students, who are familiar with solving equations and algebraic syntax, would use algebraic strategies to solve complex problems, but this was not the case. Khng and Lee (2009) already noted that many students return to arithmetic strategies in problem solving even if it is explicitly stated that the problem should be solved using equations. They found that using algebra is a step forward to higher mathematics and that students should practice algebra even if they know how to solve a problem using arithmetic strategies. In this context, the persistence in using arithmetic strategies could be seen as an inhibition for further learning. Therefore, we could pose the question of whether the algebraic knowledge of 6th graders is only formal since students did not see the equation as a suitable model for solving problems with complex structures. Students also did not use geometric models or visual representations for representing problems, which implies that they are not eager to use them in problem solving.

7. Conclusion

It is essential to understand the obstacles that students face at each level of education when solving compare-combine word problems and suggest ways to overcome them. Findings form the literature presented in the theoretical part of the paper imply that students have difficulties in understanding the simple compare word problems with inconsistent formulations. Our research showed that the difficulties are present also on the problems with more complex structure that are generated by integrating compare and combine problems. Considering that these problems also reflect students' understanding of mathematical terminology and situational understanding, they serve as a foundation for solving more complex routine and non-routine problems in mathematics education, making it essential for students to solve them correctly and efficiently, without relying on superficial strategies.

In our study, we analyzed the use of superficial strategies in solving compare-combine word problems with varying correspondence between entity order and order of description. Interestingly, we found no significant differences in problem solving with respect to the correspondence. Students often used a "shortened" model by converting the problem text into a mathematical model, which involved choosing an arithmetic operation based on a quick and superficial analysis of the data, relying on keywords in the text.

Interestingly, younger students (2^{nd} and 4^{th} graders) showed a higher association in achievement on problems with varying correspondence than older students (6^{th} graders). Students in 2^{nd} and in 4^{th} grade used superficial strategies (keywords for operations and entities) more frequently than 6^{th} grade students. On the other hand, regardless of the students' age, incorrect relational terms were the most frequent type of mistake made. Thus, the main obstacle in solving compare-combine word problems was the consistency effect – the use of the opposite operation due to a misunderstanding of relational terminology.

Based on our findings, we suggest that instructions aimed at developing a conceptual understanding of relations should be incorporated into the curriculum starting from the first grade. Conceptual understanding includes perceiving the structure of compare-combine word problems and understanding both consistent and inconsistent formulations. The literature offers three possible guidelines for improving students' achievement. First, Boonen and Jolles (2015) showed that instructions directed at developing the meaning of relations can eliminate the consistency effect. Second, a series of research is focused on the advantages of graphically representing the structure of the problem using diagrams (Boonen & Jolles, 2015; De Koning et al., 2022). These representations could be used for improving the understanding and achievement of students and reducing the consistency effect on higher levels of education. The use of representations for solving problems is related to the third guideline – the use of phases of mathematical modeling. Numerous studies suggest that students do not use phases of mathematical modeling during the problem-solving process (Stern & Lehrndorfer,1992; Blum & Leiss, 2007). The results of our study support the idea that word problem solving with the use of modeling process should be included in the curriculum in the early years of mathematics education. The ability of students to understand the problem situation and construct relations between elements of the problem determines their success and eliminates the use of superficial strategies in problem solving.

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Površinske strategije u rešavanju tekstualnih zadataka poređenja-kombinovanja

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Apstrakt. Zadaci poređenja i kombinovanja se koriste u osnovnoškolskoj matematici kao deo uobičajene nastavne prakse. Njihova integracija omogućava kreiranje više tipova tekstualnih zadataka različite strukture i nivoa složenosti. Jedna od glavnih prepreka u njihovom rešavanju je upotreba površinskih strategija u kojima učenici direktno prevode reči koje su prepoznali kao entitete i odnose, u matematičke operacije i izraze, bez razumevanja situacionog modela problema. Cilj ovog rada je da istraži upotrebu ovih strategija u rešavanju integrisanih zadataka poređenja-kombinovanja. U tu svrhu kreirali smo zadatke sa različitim korespondencijama između ključnih reči entiteta i odnosa datih u tekstu zadatka. U istraživanju je učestvovalo 134 učenika koji su rešavali zadatke u formi papir-olovka, od kojih je 44 učenika bilo u 2. razredu (učenici od 7,5 do 8,5 godina starosti), 48 u četvrtom (učenici od 9,5 do 10,5 godina), a 42 u 6. razredu (11,5 do 12,5 godina). Rezultati su pokazali da učenici nisu imali različita postignuća u tekstualnim zadacima sa različitom korespondencijom između ključnih reči i odnosa datih u tekstu zadatka. Površinski pristup koji su učenici najčešće koristili bio je u identifikaciji relacionih pojmova (matematičke operacije). Očekivano, postojale su razlike u postignuću i prirodi grešaka u zavisnosti od stepena obrazovanja učenika (2., 4. ili 6. razred).

Ključne reči: tekstualni zadaci poređenja, tekstualni zadaci kombinovanja, strategije rešavanja problema, tekstualni zadaci

On the Experience of a Problem-Posing Activity with Second-Grader Primary School Pupils

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Abstract. Representing quantities by numbers and describing the relationships between them using mathematical operations and relationships, in short, developing quantitative reasoning skills, is one of primary school mathematics education priorities. As part of a broader research project based on problem-oriented curriculum development, we are now investigating how a problem-oriented approach contributes to the development of quantitative reasoning. Our research focuses on authentic text-based problem-solving and problem-posing. In this presentation, we analyze the classroom activities of 2nd-grade primary school pupils to answer two questions. Is problem-posing as a teaching method in mathematics classrooms applicable at the age under study? Does the problem-posing activity fulfill its role of providing information about the level of understanding of the mathematical concepts being taught? To answer these questions and draw pedagogical conclusions, we aim to analyze the video recording of the lesson planned by the research team.

Keywords: problem-posing, quantitative reasoning, problem-oriented teaching approach, primary school students, multiplication and division problems

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1. Introduction

During active learning, students are active contributors, taking responsibility for their ideas and connecting them through analysis, synthesis, and evaluation (Gogus, 2012). According to Pólya (1962), learners learn by doing. An essential action in learning is to discover something on your own. One of the fundamental tasks of the teachers, as opposed to the teacher-centered approach to education, is to help their students in the process of discovery. The problem-based approach to mathematics education is part of the active learning paradigm. The application of problems in mathematics education can be achieved through different teaching strategies. In this paper, we use the approach described by Csíkos (2010) to understand problem-based learning as (1) asking students to analyze mathematical problem situations, (2) critically approaching their own and their peers' ideas, and (3) explaining and justifying their reasoning. This learner-centered, constructivist approach obviously has its place also in the early years of schooling. However, the question arises as to which learner-centered teaching methods and tools can be successfully applied at 6–9 years old.

As part of a broader research project based on problem-oriented curriculum development, we investigate how a problem-based approach can be implemented in a second-grader mathematics classroom. Recent research focuses on authentic text-based problem-solving and problem-posing, aiming to answer the question: Is problem-posing as a teaching method in mathematics classrooms applicable at the age under study? Analyzing the students' problem-posing products, a second question is also raised: Does the problem-posing activity provide information about the level of understanding of the mathematical concepts being taught? If the answer to the question is positive, it can provide an argument in favor of the problem-posing activity application, even in early grades.

2. Theoretical background

A problem-level task is one in which the starting state and the goal to be achieved are known, but the path leading to it is unknown to the student (Pólya, 1962). In addition, Pehkonen (1997) distinguished between closed and open problems. A problem is closed if the starting situation and the goal are explained precisely, while a problem is open when the starting situation and/or the goal are not precisely described.

Nowadays, there is a strong research interest in mathematical problem-posing (Koichu, 2020; English, 2020). Silver's (1994) traditional approach to problem-posing includes inventing new problems based on individual situations and reforming existing problems. Arguments in favor of problem-posing as a student activity refer to the nature of mathematical thinking on the one hand and consider it as one of the possible means of implementing active learning on the other hand. (Cai, 1998; Silver & Cai, 1996; Silver, 1994). Silver also states a relationship exists between the posed problem's quality and the individual's mathematical knowledge.

Furthermore, in generating problems based on the mathematical ideas and relations embedded in situations, students engage in "mathematizing" those situations.

In the early 1990s, Silver (1994) outlined the perspectives on mathematical problem-posing, determining some directions of problem-posing research as follows: Problem-posing as

— a feature of creative activity or expectational mathematical ability,

— a feature of inquiry-oriented instruction,

— a prominent feature of mathematical activity,

— a means of improving students' problem-solving,

— a window into students' mathematical understanding,

— a means of improving student disposition toward mathematics.

Our research highlights the "window role" of problem-posing activities, i.e., the insight into students' mathematical thinking and their mathematical experiences.

Stoyanova and Ellerton (1996) classified problem-posing situations into three categories. The situation is free when the teacher asks the students to formulate a problem based on a given, imagined, or real situation. It is semi-structured when the teacher gives the students an open-ended situation and encourages them to explore the structure and complete the problem. Finally, they consider the situation structured when the teacher encourages the students to explore a specific problem, solve it, explore possible connections between the problem statement and the solution idea, and finally, create a new problem based on the previous ones. Involving Pehkonen's categories (1997), we can establish that in free problem situations, both the starting point and the goal are open. In a semi-structured situation, the starting point is open, while the end is closed. A structured situation is characterized by the fact that the starting point is closed while the end is open.

In Hungary, Tamás Varga (1987) was the researcher who emphasized the importance of open everyday situations. In open situations, students recognize and formulate mathematical tasks. This type of problem-posing helps them apply the acquired mathematical knowledge. Kovács et al. (2023) reported a teaching experiment focusing on structured problem situations that support problem-oriented curriculum processing in 6th graders' mathematics classrooms. The present study also relates to this research and Varga's idea of open situations; we investigate the possibility of implementing problem-posing activities in learning arithmetic and quantitative reasoning with 2nd graders.

Representing quantities by numbers and describing the relationships between them using mathematical operations and relationships, in short, developing quantitative reasoning skills, is one of primary school mathematics education priorities. The basic concepts in teaching and learning elementary mathematics are numbers with their analytical and representational meaning (Nunes et al., 2016). Numbers are signs for quantities. Representational meaning refers to using numbers to represent quantities, while analytical meaning is defined by a number system. Arithmetic is based on relations between numbers, while quantitative reasoning is based on relations between quantities. Two types of quantitative reasoning are distinguished: additive and multiplicative (Figure 1). Now, we focus on the numbers from 0 to 100, the multiplicative operations, and multiplicative reasoning. There are different situations, namely schemas, where multiplicative reasoning can appear.

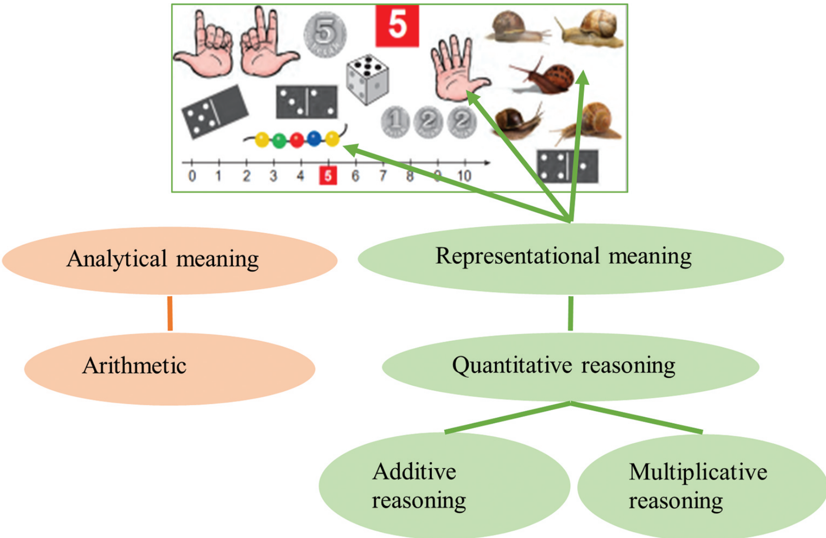

Figure 1. Quantities and their relationship.

Situations that involve a direct ratio between two quantities when the unit value is given are multiplication, partitive division, and quotative division (Nunes & Csapó, 2011). Multiplication is often introduced as the addition of repeated terms. We illustrate these operations with an example: "We put 24 colored pencils in boxes. We put 6 in each box, so we need 4 boxes to put the pencils in." These problems are solved by multiplication, partitive division, or quotative division, depending on which quantity (24, 6, or 4, respectively) is unknown.

We argue that problem-posing activities can contribute to developing students' understanding of the meaning of arithmetical operations and quantitative reasoning.

3. Methodology

We intended to implement problem-oriented mathematics teaching in line with the currently valid Hungarian curriculum through a teaching experiment involving a group of twenty regular students in the second grade of lower elementary school (7–8-year-old children). The experiment was carried out at an urban school in Hungary. The experimental lesson planned by the research group was led by a teacher with several years of professional experience who has also been involved in the practical training of pre-service teachers for many years. The lesson's topic was the comparison of multiplication tables 2, 4, and 8. The teacher created a problem situation in which students were encouraged to pose problems independently. The participating students had not had this type of lesson previously. It was also the teacher's first implementation of a problem-posing activity in her class.

The lessons were videotaped, and the episodes dealing with problem-posing were transcribed.

4. Results and discussion

This section highlights three learning episodes: creating the problem situation, setting the problem-posing task, and sharing the problem-posing products with the class.

4.1. Creating the problem situation

The problem situation was introduced with a well-known Hungarian children's song. This song is popular among kindergarten and primary school children, and the pupils were happy to sing it together with the teacher in the class. The song is about the end of summer when the migrant birds set out on their journey, and "Varjú Varga Pál" makes shoes for each bird. However, the tit is sad because she does not have shoes.

Before singing the song, the teacher set the following math problem, thus creating the problem situation:

> Teacher: "I'll tell you the task. Listen to the music, pay attention to how many birds Varjú Varga Pál has made shoes for, and if you know how many birds, then try to calculate how many shoes he made in total. ... How many legs do birds have?"

After discussing the solution (there are 4 birds in the song, so they need 8 shoes), the teacher initiated to answer to the following question:

> Teacher: "But the poor tit's feet were also very cold, so Varjú Varga Pál acquired leather for 8 more shoes. How many tits can he make shoes out of?"

A little later, she extended the question to also cats and spiders. Based on the projected images (Figure 2), it was discussed that tits have 2 legs, cats have 4 legs, and spiders have 8 legs. The solutions were recorded in a spreadsheet by the children. This task was used as a model problem for the following problem-posing activity.

The problem was authentic and motivating for young children because they could imagine themselves in fairytale situations; for them, it was close to reality. The students were eager to get involved; moreover, the song established the link between math, music, and science school subjects.

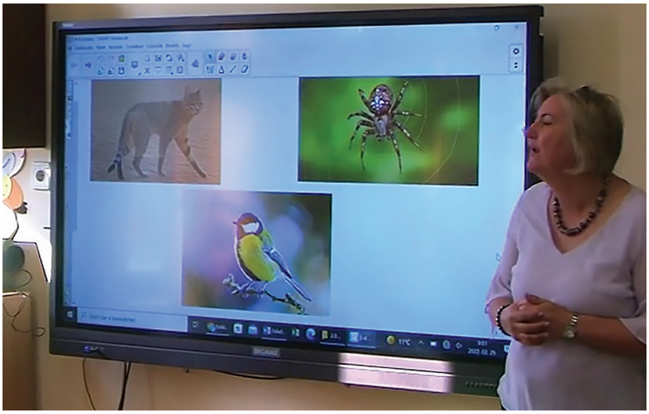

Figure 2. How many legs do cats, spiders, and tits have?

The discussion revealed that the students correctly applied the multiplication operation; even the answer could be done using division. Moreover, quantitative reasoning also appeared.

Teacher: Why can he make shoes for 4 tits, 2 cats, and 1 spider?

Student 1: Because 4 times 2 equals 8, 2 times 4 equals 8, and 1 time 8 equals 8.

Teacher: Which of them are the most that can get shoes?

Student 2: Tits get the most.

Teacher: Why?

Student 2: Because the tit has the fewest legs, it only has two.

4.2. Setting the problem-posing task

The teacher's instruction was ambiguous on the first try, resulting in a free problem situation (Stoyanova & Ellerton, 1996). The answer of Student 3 reflected this very well.

Teacher: "Now, your task will be to come up with questions about animals' legs."

Student 3: I think cats have four legs because they don't fly.

When the teacher realized that her instruction was not what she had intended, she tried to be more specific, referring to the previously solved task. In this way, she switched to a structured problem situation.

Teacher: "But I was just wondering if we could ask each other some math questions about animals . . . What did we do before? What questions did I ask on the worksheet?"

One of the students recalled the task, and the teacher tried to clarify what kind of problems she expected.

Teacher: "That's what it's about. We talked about how many shoes each animal needs. Could someone ask such a question? About animals. Can you ask a question?

Student 4: The tit holds on with its feet... how many toes does it have?...

Yelling: I know!

Teacher: Well, let's ask something we can answer.

Student 5: How many claws does the cat have?

Student 6: I know, I know, ... 4.

Student 5: No, there are 5 claws.

Teacher: Let's ask more.

Student 6: How many shoes do 8 tits have?

The teacher's latest attempt was not perfect either, but the students slowly figured out the type of questions she expected them to ask.

Even though the initiation of the problem-posing activity in this episode did not go as the teacher would have liked, it still provides several findings. First, it highlights the unusual request for students to ask questions they know the answer to. Second, the non-mathematical questions posed by the students illustrate that problem-posing activities can be a natural way to create opportunities for integrated mathematics and science (or even STEM) education.

4.3. Problem-posing products and answers

Pairs of students took turns to ask and answer each other's questions (Figure 3). After the pair work, the teacher allowed five volunteers to ask a question in front of the class and choose a classmate to answer.

Figure 3. Problem-posing in pairs.

The mathematical problems posed by these students, the answer, and the method of solution are summarized in Table 1.

Table 1. Problem-posing products and answers.

Question		Answer	Why? (justification)	Operation required by the question
Q1	How many shoes do 8 tits have?	16	I reversed the multiplication and added 8 twice.	multiplication, 8×2
Q2	How many tits have 12 shoes?	6 tits	1 tit has 2 shoes, 6 tits have $6 \times 2 = 12$ shoes.	quotative division, $12 : 2$
Q3	How many shoes are there for 8 spider's feet?	64	Well, [I counted it] simply backward from 80 [80, 72, 64], and it [the arithmetic operation] is $8 \times 8 = 64$.	multiplication, 8×8
Q4	There are 48 spiders. How many shoes do spiders have?	6 spiders (incorrect)	Because $6 \times 8 = 48$.	multiplication, 48×8 (incorrectly quotative division, $48 : 8$)
Q5	How many shoes can fit on 15 cats?	60	$15 \times 4 = 60$	multiplication, 15×4

Even though the model problem was a quotative division problem ("Varjú Varga Pál acquired leather for 8 more shoes. How many tits can he make shoes out of?"), all but one (Q2) student posed a multiplication problem with almost the same context. However, in all cases, the calculation was done by multiplication or repeated addition. The answer to Q3 shows that multiplication is understood here as repeated addition since the following strategy is used: $8 \times 8 = 10 \times 8 - 8 - 8$, which is an efficient implementation of the $8 + 8 + 8 + 8 + 8 + 8 + 8 + 8$ calculation.

From this problem-posing episode, we can observe that

— the meaning of multiplication is already well understood, but that of division is not.

— the multiplication problem is often solved by repeated addition or even subtraction.

— the division problem is solved as an incomplete multiplication.

We can conclude that students at this age are able to pose problems based on the model problem. In the introductory model problem, the question concerns the number of tits, cats, and spiders. The word problems posed by the students included these animals and the number of shoes on their feet. The mathematical structure of the posed tasks is very similar, and they are more likely to lead to multiplication than division. It can suggest that the students are not yet confident with division.

In the rest of the lesson, the number of shoes needed was calculated according to the number of cats, spiders, and tits. They tabulated the data, looking for relationships between the elements of the multiplication tables 2, 4, and 8 on the arithmetic frame.

5. Summary

The use of the framework story provided by a children's song guided the lesson well, and the principles of the problem-based lesson were implemented in the context of problem-posing and solving based on a model problem. The students supported their ideas with arguments, although these arguments were brief and mainly in response to a teacher's question.

We reflect our research questions as follows:

— Is problem posing as a teaching method in mathematics classrooms applicable at the age under study?

Students of Grade 2 can pose mathematical problems. These are strongly linked to the model problem in terms of context and mathematical content. However, the instructions "Ask a question." or "What are you curious about?" were initially misleading, especially with the addition of "Ask me something you know the answer to!". Children in everyday situations usually ask questions they do not know the answer to. Therefore, initiating an appropriate problem-posing activity requires precise and well-directed instruction from the teacher.

— Does the problem-posing activity provide information about the level of understanding of the mathematical concepts being taught?

Even at this initial stage, problem-posing activities can be used to gather information about the current level of children's thinking. With one exception, the posed problems could be directly solved by multiplication, or if not, the pupils used incomplete multiplication instead of quotative division. Furthermore, students often applied repeated addition to answering the multiplication question. This shows that they know the meaning of multiplication but are still unsure about division. The multiplication tables are only seldom applied because they feel still more confident in additive thinking.

During the classroom discussion, it can be observed how second-grade children reason, how they put their own train of thought into words, and how they are able to express relationships between quantities through operations. Quantitative reasoning already appears in the newly learned multiplicative operations. Developing this ability further is a priority for primary schools.

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Tapasztalatok a másodikos tanulók problémaalkotási tevékenységéről

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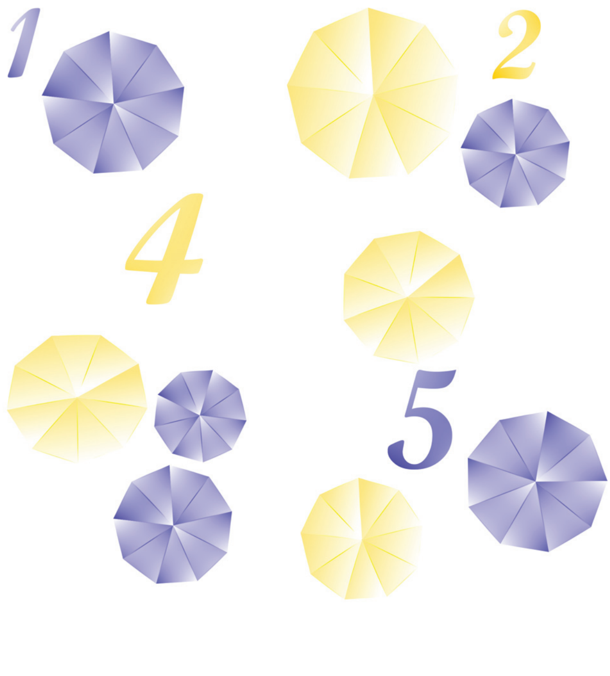
Absztrakt. A mennyiségek számokkal történő reprezentálása, valamint a mennyiségek közötti kapcsolatok leírása matematikai műveletekkel és relációkkal, röviden a mennyiségi érvelés képességének fejlesztése, az általános iskola alsó tagozatának kiemelt feladata. Egy problémaközpontú tananyagfeldolgozással foglalkozó átfogó kutatási projekt részeként most azt vizsgáljuk, hogy a problémaközpontú megközelítés hogyan járulhat hozzá a mennyiségi érvelés fejlődéséhez. Kutatásunk az autentikus szöveges feladatokon alapuló problémamegoldásra illetve a problémaalkotásra irányul. Ebben az előadásban második évfolyamos alsó tagozatos tanulók osztálytermi tevékenységét vizsgálva két kérdésre keressük a választ: Alkalmazható-e a problémaalkotás mint tanítási módszer a vizsgált korosztályban? Betölti-e a problémaalkotás azt a szerepét, hogy információt nyújt a tanított matematikai fogalmak megértésének aktuális szintjéről? A kérdések megválaszolásához és a pedagógiai következtetések megfogalmazásához a kutatócsoport által tervezett tanóráról készült videófelvételeket elemezzük.

Kulcsszavak: problémaalkotás, mennyiségi érvelés, problémaközpontú tananyagfeldolgozás, alsó tagozatos tanulók, szorzási és osztási problémák

2.
Development of Mathematical Thinking through Theory, Modelling, and Activity



The Potential of Number Theory in the Development of Mathematical Thinking

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Abstract. Number theory is a small part of the Hungarian secondary school curriculum. Number theory is an area of mathematics where the ideas used are different in many ways, and where the types of problems are varied. Students can encounter a wide range of problems, from simple basic problems to Olympic competition problems. This is why we started to investigate how number theory tasks can affect the development of students' thinking and general mathematical reasoning in public schools. We designed an experiment that links the learning of number theory and the solving of number theory problems with general mathematical problem-solving skills.

Seventh and eighth grade students participated in the experiment. The students in the experiment were divided into two groups, an experimental group and a control group. The students in the experimental group were given a number theory task at the beginning of each lesson, while the students in the other group were given a task related to the regular curriculum. The effectiveness of the experiment was tested on both groups by regular curriculum assessments and by levelled assessments in which number theory was not included.

The results confirm that students who have practised number theory tasks over several months perform better on the regular curriculum. We can conclude that our method was effective in the medium and long term.

Keywords: problem solving, number theory, secondary school, Case Study, classroom settings, classroom routine

1. Introduction

In general, the extent to which human thinking develops, as well as problem-solving ability, depends on a number of factors, such as the quality and type of outside help (Beaten et al., 2008), the type of the task that is solved, and what kind of topic is involved. The secondary school mathematics curriculum covers a very wide range of fields, so teachers can design a variety of tasks and are free to choose the tasks they wish to use. Yet, when we teach, we usually only deal with one subject at a time, and there is no opportunity to develop students' skills with varied tasks in problem solving. In the Hungarian secondary school curriculum, number theory appears as a marginal part. Some European countries do not teach this subject at all, for example Germany. Number theory is the area of mathematics where we find the most diverse ideas and the most varied types of problems, from simple classroom problems to Olympic competition problems.

2. Number theory education in Hungary

Number theory is a theme that follows throughout primary school. Hungarian pupils are introduced to the terms even and odd numbers, multiples, divisors and remainders already in primary school. In grade 5 they are already learning about addition, subtraction, simplification and expansion of fractions. When adding and subtracting fractions, the ability to find the least common denominator is required, which can lead to a prior knowledge of the definition or determination of the least common multiple. Simplifying fractions then leads to the term of the greatest common divisor. Above the age of seven, everyone is capable of learning abstract concepts presented at the right level of difficulty and in the right language (Bruner, 1978; Mcleod, 2023), and if we build the school curriculum according to Bruner's principle of spirality (Wood et al., 1976; Mcleod, 2023), thirteen-year-old students will be mature enough to formulate their experiences in general ways. In the 7th and 8th grades, they extend the division rules that they have learned before. They repeat prime factorisation and the method of choosing the greatest common divisor and least common multiple using non-negative integer powers of positive integers. In parallel, at the end of primary school, we should help them develop the need for proofs (Senk, 1989), so that they are later able and interested in using conjectures, theorems and definitions to solve problems and explain their solutions. Contrary to primary school, where number theory does not seem to be a marginal part of the curriculum, in secondary school it is almost neglected, apart from a few lessons in year 11. Until 2020, the topics of number theory were built into the ninth-grade curriculum. The teacher may decide to repeat the subject briefly in year 12 before the final exams, but this means only a few extra lessons. The simple reason for this may be that there are few number theory exercises in both the intermediate and advanced levels of the final exams, and most of them can be solved with primary school knowledge. Despite this, there is no national mathematics competition without a number theory problem every year. Secondary school number theory is a field of mathematics where, apart from some basic knowledge, not much lexical knowledge is needed. So the most important thing you need for successful problem solving is clarity and elementary thinking, which is why it is an inevitable part of mathematics competitions. We therefore suspect that this topic could also be a suitable way of developing students' general problem-solving skills. As in Hungary, the development of problem-solving skills is a key competence in mathematics education in many countries. Therefore, its development has been studied from many perspectives (Pehkonen, 2013; Kovács & Kónya, 2019). In our research we highlight problem solving as one of the key competences of the Hungarian National Core Curriculum. In Hungarian mathematics education, we focus on the development of problem recognition and problem-solving competences.

Our experiment is a case study conducted in 2019–2020. The experiment was not originally designed as a case study, but the COVID pandemic intervened and changed the experimental conditions to allow only online teaching.

3. The method

We wanted to test the long-term effect of the experiment, so it was important to collect enough tasks for at least three months. Our concept was to collect 15 to 20 competition tasks, ranging from easy to the difficult ones. Once we had selected these, we created guided exercises depending on the difficulty, with 1 or 2 exercises for the easier ones and 3 or 4 for the harder ones. The chosen competition tasks were collected into four blocks depending on the topics, and four blocks were also created within the task set. Each block is based around some specific number theory ideas and terms. For example, the problems related to the greatest common divisor are grouped in the first block, while the problems related to the parity of primes are grouped in the fourth block. We also created an order between the problems within each block. On the one hand, the competition tasks in a given block were ranked in order of difficulty, and on the other hand, the corresponding 2-3-4 guiding tasks were inserted before each competition task. This gave a total of 63 tasks. The exercises only build on primary school knowledge, so we can imagine that they could be suitable for more than one age group. In our experiment, the participants were 8th graders.

When designing our experiment, it was important to keep the students motivated and to ensure that they actually participated in the experiment. In Hungary, many students are not self-motivated enough to acquire knowledge, therefore getting good grades is a motivation to learn. To keep the students active and motivated, they collected points throughout and were awarded 5s at the end of each block.

The participants in the case study were a class of eighth graders, divided into two groups for the experiment, one half becoming the experimental group and the other half the control group. In order to prevent students from knowing which group they belonged to, we introduced a seemingly similar class routine for the two groups. At the beginning of each lesson, the participants of both groups had to solve a task, which was collected by the teacher after 5–10 minutes. While the experimental group solved a problem on number theory, the control group solved a problem on the topic the class had just learned. Overall, students in the control group spent 20–25 minutes more per week on the regular curriculum than students in the experimental group. The method is designed to make the students feel comfortable in the classroom during the experiment and to make problem-solving at the beginning of the lesson a kind of classroom routine (Newell & Orton, 2018; Sword et al., 2018).

We graded students' work to make sure they took the tasks carefully and with motivation. In order to support students who find it more difficult to set the tasks at the beginning, we decided to point collection – that is, we gave the students points for solving each problem, rather than a grade. The tasks were divided into four parts and the points added up at the end of each part. The students who achieved a certain target point total at the end of each section were awarded the best mark in the Hungarian system, a 5. This usually meant 50 % of the points. This gave students who did badly at the beginning a chance to get a good grade, while those who did well got more fives. One of the aims of our study was to give students as rich ideas as possible, meaning that most of the points awarded for the submitted number theory problems were for the appearance of good ideas, not for solving them. In this way, we communicated to the students that they should invest energy in writing down the details and that they should be aware that in mathematics, ideas are often more important than the final result. This was explained to the students before the first exercise, and also during the process, that it is not enough to write down the result. A spreadsheet was available for students to check their scores achieved. Meanwhile, the control group solved tasks related to the current curriculum. They also took part in the point collection. The tasks were always worth the same number of points for both groups.

4. Results

The students worked on the experiment for twenty weeks. The effect of the study was assessed by regular curriculum tests and levelled tests that did not include number theory. In total, they wrote four tests during the experiment.

Five students in the class achieved the highest score on all the tests they wrote during the year. Of the five students, three belonged to the control group and two to the experimental group. They are omitted in the analysis of the results, as they were considered to distort the effect of the experimental method because, contrary to the other students, they were very hard working. We gave the scores and averages on each test in this assumption.

At the beginning, we assessed the prior knowledge of the students in the experimental and control groups and examined whether the two groups had different mathematical abilities. To do this, we collected the results of the first final exams written by students in the same class at the beginning of the school year (Input Test). Each class wrote this when the classes had already been divided within the experiment (experimental and control groups), but had not yet started to solve the problems of the experiment. This Input Test was written on the topic of algebra.

The second test (Functions 1) was written in week 15 of the experiment on the topic of functions. The third test (Levelled Assessment) was written one week after the last number theory problem (on the last lesson of the experiment) was submitted. This third test included a selection of exercises on topics that they had studied this school year but which are not related to the topic of number theory. The fourth test (Functions 2) was written 3–4 weeks after the last number theory question on the topic of functions. We chose functions because we wanted to assess the students' knowledge of a topic that they had studied 2–3 months before. Thus, the topic would not be new but one that the students should still remember. By doing this, we hoped to also look at the long-term impact on their knowledge.

For the first test at the beginning of the year ("Input Test"), students could score a maximum of 40 points. The results of the test are shown in Table 1.

Table 1. The results of the Input Test.

Experimental group	40	39	39	39	38	37	36	33	M: 37,63
Control group	39	37	36	34	34	30	–	–	35

The table shows that the difference between the two groups' performance is minimal, so the two groups can be considered to be equal. In the 15th week of the experiment, they wrote a final test on the topic of functions (Functions 1) with a maximum score of 31 points. These results are shown in Table 2.

Table 2. The results of the second test (Functions 1).

Experimental group	31	30	29	29	28	28	28	27	M: 28,75	D: 1,28
Control group	29	28	27	25	21	21	–	–	M: 25,17	D: 3,49

A week after the last number theory task, they took the Levelled Assessment. This paper consisted of regular curriculum tasks with a maximum score of 24 points. The results are shown in Table 3.

Table 3. The results of the third test (Levelled Assessment).

Experimental group	24	24	24	24	23	21	20	17	M: 22,13	D: 2,59
Control group	24	20	19	16	14	9	–	–	M: 17,00	D: 5,22

The fourth test (Functions 2) was written four weeks after the last number theory problem. The maximum score on this paper was 35 points. The results are shown in Table 4. On this test, there were four tasks, two of which were of average difficulty (Tasks 1 and 3) and two of which were more difficult and required more thinking (Tasks 2 and 4). While the first and third problems involved easier function diagrams, the second and fourth problems were more complex. Complexity not only in terms of the number of steps used during the solution, but also in the fact that algebra and geometry approach helped a lot while solving.

The results obtained are detailed for each task. For ease of reference, the results of the experimental group are highlighted in red and those of the control group in green.

The maximum score on the second test (Functions 1) was 31. The mean score for the experimental group was 28.75 with a standard deviation of 1.282, and for the control group the mean score was 25.167 with a standard deviation of 3.488. The maximum score possible on the Levelled Assessment was 24. The mean score for the experimental group was 22.13, with a standard deviation of 2.588, and the mean score for the control group was 17.0, with a standard deviation of 5.215. On the fourth test (Functions 2), the maximum score available was 35. The mean score for the total score was 29.5 with a standard deviation of 2.726 for the experimental group and 22.67 with a standard deviation of 6.154 for the control group.

Table 4. The results of the fourth test (Functions 2).

Task	1.		2.		3.		4.		∑	
Max point	10		11		8		6		35	
	10	10	11	6	8	8	6	6	33	28
	10	10	10	6	8	8	6	4	32	27
	10	10	9	5	8	8	6	3	30	25
	10	10	8	3	8	8	6	2	30	23
	10	9	6	3	7	7	6	0	30	22
	10	9	5	1	7	0	5	0	29	11
	9		5		7		5		28	
	8		3		5		4		24	
M	9,625	9,67	7,125	4,0	7,25	6,5	5,5	2,5	29,5	22,67
D	0,744	0,516	2,8	2,0	1,035	3,209	0,756	2,345	2,726	6,154

The students in the experimental groups scored significantly higher than those in the control groups, despite the fact that the two groups performed similarly on the Input Test.

Table 4 also shows that on the tasks that required more thinking (2 and 4), the difference between students in the experimental group and the control group is even more significant. It can also be said that all but one student in the experimental group scored at least as high as the highest performing student in the control group.

One might think that students who have practised the regular curriculum more should do better on the tests written on this material. The results recorded in Tables 2, 3 and 4 confirm that students who have practised number theory tasks over several months perform better on the regular curriculum. We can conclude that our method was effective in the medium and long term. We were not able to "properly" assess the impact of the experiment on the excellent students who learn regularly and well in the traditional way. Even without the experiment, they almost always achieve maximum results. This does not mean that they have not improved, nor that the effect of our experiment can be replaced by traditional learning. Their progress could be studied in a separate experiment.

5. Discussion

We designed an experiment that links the learning of number theory and the solving of number theory problems with general mathematical problem-solving skills. Eighth grade students participated in the experiment. The results of 14 students were analysed in detail, while 5 students were not included due to learning conditions. The students in the experiment were divided into two groups, an experimental group and a control group. The students in the experimental group were given a number theory task at the beginning of each lesson, while the students in the other group were given a task related to the regular curriculum. The effectiveness of the experiment was tested by regular curriculum assessments and by levelled assessments in which number theory was not included.

The students worked on the experiment for twenty weeks. In total, they wrote four tests during the experiment. The students in the experimental groups scored significantly higher than those in the control groups, despite the fact that the two groups performed similarly on the Input Test. On the test written four weeks after the experimental lessons, we examined the tasks separately and we found that for the two tasks requiring more thinking, the difference between the two groups was even greater in favour of the experimental group.

Our experiment is a case study, conducted in 2019–2020, before the COVID epidemic. In 2020, the National Core Curriculum was rewritten in Hungary. For this reason, the topic of number theory was moved from grade 9 to grade 11 in the high school. Based on this, we consider it necessary to set up an experiment similar to this one, involving ninth grade students. We believe that number theory is a subject that develops mathematical thinking in general and that it is therefore not at all irrelevant when it is introduced during the 4 years of secondary school. Nor is it indifferent in terms of forgetting graphs and developing thinking as effectively as possible.

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A matematikai gondolkodás fejlesztése számelméleti feladatokkal

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Absztrakt. A magyarországi középiskolai tananyagban a számelmélet témaköre kis súllyal szerepel. A számelmélet a matematikának az a területe, ahol a legkülönbözőbb ötletekkel, legváltozatosabb feladattípusokkal találkozunk az egyszerű órai feladatoktól az olimpiai versenyfeladatokig. Ezért kezdtük el vizsgálni, hogyan hathatnak a számelmélettel kapcsolatos feladatok a közoktatásban tanuló diákok gondolkodásának fejlődésére, általános matematikai gondolkodására. Egy kísérletet terveztünk, amely a számelmélettel való ismerkedést, számelméleti feladatok megoldását összeköti az általános matematikai problémamegoldó képességgel. A kísérletben hetedik és nyolcadik osztályos tanulók vettek részt. A kísérletben szereplő diákokat két részre osztottuk, egy kísérleti és egy kontroll csoportra. A kísérleti csoport diákjai minden óra elején egy számelmélettel kapcsolatos feladatot oldottak meg, a másik csoport tagjai a reguláris tananyaggal kapcsolatosan kaptak feladatot. A kísérlet eredményességét a reguláris tananyaggal kapcsolatos dolgozatokkal, és olyan szintfelmérőkkel vizsgáltuk, melyekben a számelmélet témaköre nem szerepelt.

Az eredmények azt mutatják, hogy azok a diákok, akik az órák alatt számelméleti feladatokkal is foglalkoztak, jobban szerepeltek a reguláris tananyagból írt dolgozatokon. A kísérletben szereplő módszer mind közép, mind hosszútávon eredményesnek bizonyult.

Kulcsszavak: problémamegoldás, számelmélet, középiskola, esettanulmány, órai rutin

Fermi Problems as a Mathematical Modeling Activity in Secondary Education

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Abstract. This paper shows how a group of seventeen-year-old students from a Croatian high school solved two Fermi problems that required estimating the number of people on a given surface. The main goal was to investigate the didactic potential of a set of Fermi estimation problems as an open mathematical modeling activity in secondary education. The models that students constructed in individual and small group work are characterized, and how the models evolved with respect to the constraints imposed by the realistic context in the different tasks is analysed.

Keywords: mathematical modeling, secondary education, open task, Fermi problems, students' strategies

1. Introduction

Mathematical modeling is an extremely important but complex process, and it is sometimes difficult to find room for it in crowded curricula. The teacher plays a key role in selecting tasks and matching them to students. When selecting modeling tasks, one can opt for closed-ended task types that focus on specific knowledge and make it easier for the teacher to guide and control the tasks, or open-ended activities that provide greater cognitive challenge for students. One of the advantages of open-ended modeling tasks is that they allow students to investigate the features of a particular phenomenon and construct an accurate mathematical model that enhances their understanding of the underlying reality. However, it is difficult for the teacher to anticipate all the possible interpretations, strategies, and mathematical concepts that students will use in creating their mathematical model. It is also important to ensure that students have enough time to investigate open-ended problems. Therefore, these types of problems are rarely presented in class and are more often assigned as independent or group projects as homework (Gusić, 2011).

The present topic is the main subject of this work. It shows how a group of 17-year-olds solved two open modeling problems that belong to the Fermi estimation problems. The problems are mathematically equivalent in that they both require estimating the number of people on a given area, but the specific context of their formulation leads to the use of different models. The strategies and models developed by students working individually and in small groups are characterized. Then, how the models change given the constraints imposed by the different task contexts is analyzed.

2. Models and Mathematical Modeling

Every time we apply mathematics to a real-world context, we begin a process of mathematization that involves the skills and problem-solving strategies that come with mathematical modeling. Authors Lesh and Harel (2003, as cited in Aymerich et al., 2017) suggest that models are conceptual frameworks used to construct, describe, or explain other systems. These models include both (a) a *conceptual structure* for representing or explaining relevant mathematical entities, relationships, actions, or patterns and (b) *the corresponding procedures* for generating meaningful constructions, manipulations, or predictions. The concept of mathematical modelling can be understood as the generation of complex configurations or systems (consisting of interconnected integrating elements) that emerge from cycles of interaction. These systems enable the understanding of reality through their simplification (Lesh & Harel, 2003, as cited in Aymerich et al., 2017).

In a mathematical modeling activity, students develop and/or use mathematical models that represent real-world situations or problems. The notion that modeling processes are inherently cyclical is widely acknowledged. Students progress through distinct stages, returning to the initial point, all in pursuit of uncovering a mathematical representation that effectively describes the studied phenomenon or offers the most fitting solution to the posed question (Albarracín & Gorgorió, 2018). The modeling cycle consists of several key steps that guide the process of creating and using mathematical models to solve real-world problems:

- Mathematization: abstracting and simplifying the real situation into a set of mathematical relations or expressions;
- Mathematical treatment: once the problem is mathematized, applying mathematical techniques, operations, and calculations to analyze and solve the mathematical representation;
- De-mathematization: interpreting the mathematical results in the context of the original real-world problem;
- Validation of the answers and model: verifying that the model's answers are relevant, useful, and applicable to the real-world problem; evaluating how well the model meets the original objectives and whether it provides valuable insights or solutions. If the model's answers meet the intended goals and match reality, the model is considered successful and effective (Niss & Blum, 2020).

The modeling cycle is not intended to outline the exact sequence of steps that each individual must follow in constructing a model, or that they must be done in the order given. The actual paths that students take in modeling can be rather complex and vary widely. Authors Niss and Blum emphasize the conceptual core of the modelling cycle as a systematic deconstruction and analysis of the essential phases of mathematical modelling:

> The modelling cycle therefore should be understood as an analytic (ideal-typical) reconstruction of the steps of modelling necessarily present, explicitly, or implicitly, as an instrument for capturing and understanding the principal processes of mathematical modelling. (Niss & Blum, 2020, p. 14)

Mathematical modelling represents two branches of education: first, as a means of teaching specific mathematical content, and second, as content in itself to encourage and motivate students to address mathematical problems from the everyday world (Julie & Mudaly, 2007, as cited in Aymerich et al., 2017). In this study, emphasis is placed on the second educational goal – the development of modeling competency.

3. Fermi estimation problems

The term Fermi problem goes back to the Italian Nobel Prize winner Enrico Fermi (1901 – 1954). Fermi was not only one of the greatest theoretical and experimental physicists of the twentieth century but is also known as an outstanding teacher (Lan, 2022). Fermi himself liked to pose and solve problems such as "How many shopping malls are there in the United States?" or "How many piano tuners are there in Chicago?". Over the years, he repeatedly posed such problems to his physics students at the University of Chicago. This was done to demonstrate the effectiveness of reasoning and to provide them with the skills necessary to conduct experiments in the laboratory. He believed that a good physicist, like any reasonable, educated person, can estimate any quantity with sufficient accuracy using only his head – by thinking, using realistic and intelligent estimates in magnitudes, and applying simple calculations. With a quick calculation, Fermi would arrive at an amazingly accurate and meaningful answer based on some reasonable assumptions and estimates. In his honor, these types of problems are called Fermi estimation problems.

Although Fermi problems were first introduced in physics, they are now used as teaching tools in many scientific disciplines. Previous classroom application studies have shown that students are able to solve complicated and practical problems. The process of solving the Fermi problems promotes the development of various skills that are essential for overcoming the obstacles of the future. They are closely related to the domain of measurement and promote estimation and number sense. Depending on the age group of the students, Fermi problems can serve as a versatile tool for conducting modeling activities that can have varying objectives. For example, at the primary education level to introduce modeling, or in secondary school to help students connect knowledge from different domains. Fermi problems placed in a cultural, social, and environmental context can teach aspects of personal and social responsibility and media literacy. They are considered a powerful tool for promoting critical thinking and aspects of general problem solving.

The focus of this paper is on the connection between Fermi problems and mathematical modeling. There are numerous reasons why the use of Fermi problems can be beneficial when introducing modeling in the mathematics classroom:

- they have a clear connection to the real world;
- they are accessible to students of different educational levels and do not require any special prior mathematical knowledge;
- they force students to define the structure of the relevant information;
- they are open-ended tasks that are not tied to any prior knowledge and require students to develop a solution strategy;
- they do not provide numerical data, students must estimate different quantities themselves;
- they stimulate discussion among students (Ärlebäck, 2011, as cited in Albarracín & Gorgorió, 2018, p. 4).

In previous research, series of Fermi problem tasks have emerged as a potentially promising research direction, as the Fermi method can be taught effectively with such an approach (Ärlebäck & Albarracín, 2019). An example of such research is the study conducted by Albarracín and Gorgorió (2018). They developed a sequence of five Fermi problems where the same problem is solved from a mathematical perspective – it is about estimating the number of people or objects located on a given area. They wanted the modeling activities to be complex and set in different contexts, close to the students but far from problems with known solution procedures. The realistic context of each task in a sequence was carefully designed to introduce new meanings, concepts, and procedures not introduced in the previous task. Students grapple with the sequence problems in a variety of conflicts, which should encourage them to gradually build mathematical models.

In their previous research, Albarracín and Gorgorió (2013, 2014, as cited in Albarracín & Gorgorió, 2018) established the following categories of strategies and models that students use and develop when solving Fermi tasks that require estimating the number of objects on a given surface:

- Exhaustive counting: the student proposes to count all the elements of a group of objects individually;
- External source: the student transfers the responsibility of solving the problem to a third party who should have the necessary information;
- Concentration measures: the student bases his decision on the value of the ratio between the number of objects distributed on a given area and the area they occupy, which he determines independently (e.g., population density – the number of people per square meter);

- Reference point: the student attempts to determine the total area of the surface on which people or objects are located and divides the obtained area by the area occupied by an object that serves as a reference point;
- Grid distribution: the student uses a mental image of the distribution of people or objects in the grid, then estimates the total number of people or objects for each dimension and applies the product rule to determine the result.

From a mathematical point of view, this could be considered the same model, but the authors point out that the models that do not have the same procedures and concepts are different models. So, from a theoretical point of view, the models listed are different. From the students' point of view, these are also different models because students usually focus more on the procedures and rarely discuss conceptual elements.

In solving the five Fermi problems, students used three different models – grid distribution, population density, and iteration of reference points. Working in groups, students gradually applied conceptually rich strategies and eventually arrived at a model – a generalization of the idea of population density – that is not only useful for solving this problem but can also be applied in other situations. It is worth noting that the study also showed that students changed their strategies and models due to procedural obstacles rather than conceptual constraints (Ärlebäck & Albarracín, 2019). The importance of Albarracín and Gorgorió (2018) study is that it opens the possibility of designing a set of modeling activities that encourage students to construct mathematical models that are relevant from a curricular perspective, while developing the skills characteristic of modeling.

4. Aim of the research

The study by Albarracín and Gorgorió (2018) was conducted in six lessons of 60 minutes each. It is not always possible for most teachers to dedicate so many lessons to a modeling activity. Therefore, the question arose whether it is possible to obtain comparable results if the given set of Fermi estimation problems is adapted and reduced (to two problems) so that it can be conducted in a shorter period (one 90-minute session).

Thus, the goal of this study is to evaluate the potential of this modified activity in the context of learning and teaching mathematical modeling at the secondary level.

Research Questions

1. What mathematical strategies and models did participants use in solving two Fermi problems involving estimating the number of people on a given area?

2. How did their strategies and models evolve when faced with the constraints imposed by a given realistic context in problem formulation?

5. Methodology

5.1. Design

A qualitative pilot study was conducted, based on the work published in 2018 by authors Albarracín and Gorgorió. A group of 19 high school students with an average age of 17 participated in the study. The students attend high school (gymnasiums program) in the city of Rijeka, Croatia, and had not encountered Fermi estimation problems in their previous education. Furthermore, estimation problems involving large quantities are not a common part of the Croatian curriculum, so it is reasonable to assume that they are not frequently encountered in the classroom. For this reason, the students with whom the study was conducted did not have ready-made specific methods for solving the given problems, so they had to develop their own strategies and models for the given situation of estimating large quantities.

The study was conducted in one session of 90 minutes duration. Given the tight time frame, students solved two Fermi problems involving estimating the number of people that can be distributed on a given area.

Problem P1: How many people can fit in your classroom if there is no furniture?

Problem P2: How many people could participate in the demonstration at Ban Jelačić Square in Zagreb?

The tasks are a variation of those designed by Albarracín and Gorgorió (2018) and were created in collaboration with the teacher. Task P1 refers to a situation that is familiar to students and can be empirically studied in school, serving as a source of strategies and models. Although the context of the second task is also familiar to the students, the location is not physically accessible to them, so the use of new (digital) tools was necessary to obtain the needed information. Figure 1 illustrates the utilization of Google Earth for this purpose.

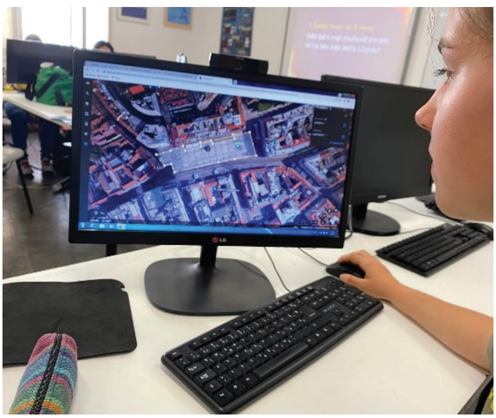

Figure 1. Application of Google Earth tool in problem P2.

In addition, Task P2 contained another contextual novelty that would influence the development of their models, namely the fact that there are areas in the square where people are not allowed to be, such as a fountain or a statue. The goal of Task P2 is to encourage students to question, develop, or reconstruct the model they created while solving Task P1 to arrive at a more versatile model. To facilitate data collection, a report form was designed to record the text of the task to be solved, a spontaneous estimate of the answer, and a plan or procedure for solving the task.

Fermi estimation problems are open-ended problems where it is usually not possible or necessary to determine an exact solution, but a sufficiently well-informed estimate of the solution is required. Since we rarely encounter a situation in class where we have different but correct solutions, I felt it necessary to familiarize students with this feature of Fermi problems. For this reason, in the introductory part of the session, two examples of Fermi estimation problems were solved on the blackboard and discussed together by the whole group. In selecting the examples, care was taken to ensure that they were simple enough to be solved relatively quickly and that they had no conceptual or procedural connections to the problems that the students would solve later. Tasks P1 and P2 were then presented.

After task P1 was assigned to individuals, a scenario characterized by self-directed learning developed. In this context, students were tasked with independently formulating and documenting a plan for solving Task P1 within a given time frame. They did not have to solve the task. After the time was up, they submitted their report; there was no collaborative discussion. The students were then randomly divided into groups of 3 or 4 and asked to solve task P1 as a group this time, explaining their working methods and the solution they found.

Measuring instruments (tape measure) and desktop computers were available in the classroom if students wanted to use them. After the allotted time, each group presented their results, and a brief discussion took place. After discussing the results, they stayed in the same groups and solved problem P2. Due to the lack of time, the students presented their results only briefly and a more detailed discussion was not held.

5.2. Data

The data collected consisted of:

- 19 individual reports on plans to solve problem P1.
- 5 reports on solutions to problem P1, which were developed in group work.
- 5 reports about solutions to problem P2, which were developed in group work.

The individual and group reports were coded. Each group is assigned a letter from A to E, and each student in the group is named by adding a number to the letter of the group to which he or she belongs.

6. Analysis and Results

We proceed with an interpretative qualitative data analysis using the model types developed by Albarracín and Gorgorió (2018) and described in Section 3. Examples of task resolutions used by students in their reports and prescribed model types can be seen in Figure 2, Figure 3 and Figure 4.

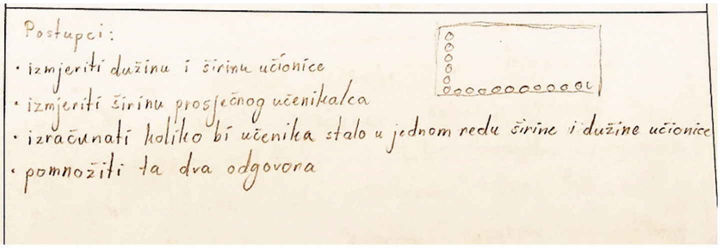

Figure 2. An example of a grid distribution model in an individual plan to solve problem P1.

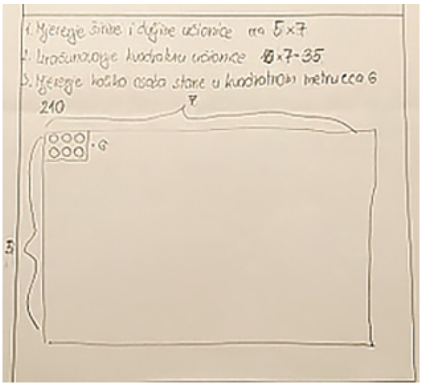

Figure 3. An example of a population density model in an individual plan to solve problem P1.

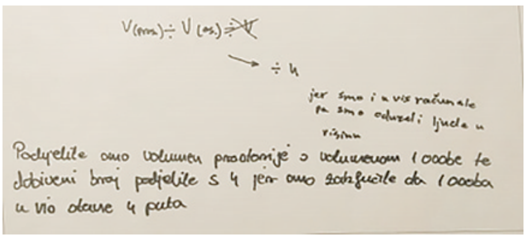

Figure 4. An example of a reference point model in a group report for problem P1 (with an incorrect estimate of the height of the classroom).

Based on the reports completed by the students, Table 1 and Table 2 show the strategies and models used. In Table 1, it can be seen that already in the individual reports where only plans to solve problem P1 were given, all three models (grid distribution, population density, and iteration reference points) were proposed and no new type of model was identified.

Table 1. Identified strategies at different moments of the P1 problem solving process.

Group	Student Individual Proposal		Group Solution of Problem P1
A	A1	Incomplete answer (Proposed to measure and determine the classroom floor area.)	Grid distribution (Estimated the number of people that fit in the row and column and multiplied the obtained values.)
	A2	Grid distribution	
	A3	Incomplete answer (Calculation only, seemed like grid distribution.)	
	A4	Incomplete answer (Same as A1.)	
B	B1	Population density	Population density
	B2	Population density	
	B3	Grid distribution	
C	C1	Reference point (RP2: Divided the classroom volume by the estimated volume of a person. Proposed to consider the volume of free space.)	Reference point RP2; Divided the quotient of the classroom volume and estimated volume of a person by the ratio between the room height and the estimated height of a person.)
	C2	Reference point (RP1: Divided the classroom floor area by the estimated area occupied by a person.)	
	C3	Reference point (RP1)	
	C4	Population density	
D	D1	Grid distribution	Grid distribution (Estimated the number of pairs of people that fit through the length and width of the classroom and multiplied the obtained values.)
	D2	Population density	
	D3	Grid distribution (Estimated the number of people in a row and divided the classroom floor area by the area of the row.)	
	D4	Reference point (RP1)	
E	E1	Reference point (RP1)	Reference point (RP1)
	E2	Incomplete answer (Proposed to determine the classroom floor area and the width of the person.)	
	E3	Population density	
	E4	Reference point (RP1)	

When solving task P1 in a group, members bring in their own previously developed strategic plans to solve the problem and must agree at the group level on which model is most appropriate. From the classroom observations and submitted reports, it can be concluded that the groups chose the model that was most represented in the individual plans of the group members.

In Table 2, we can compare the strategies identified in solving tasks P1 and P2 in groups. It can be seen that the strategies changed and that the dominant strategy for solving task P2 is population density. It is interesting to note that the idea of the population density model appeared in the individual plans of all groups, except for group A, which could not solve task P2. Students were not observed to validate their model by asking why one model was better than another. It can be assumed that the reason for changing the model was the characteristic context of task P2, which made it impossible to apply certain procedures that the groups had used in solving the previous task. In Table 2, we see that only two groups, B and C, made certain adjustments and enriched their population density model when solving task P2. Namely, they determined that certain spaces in the square were not usable, such as the area of the statue and the fountain and subtracted the area of these spaces from the total area of the square. The idea that a certain space is not usable also came to group C when solving the previous task, since they used a reference point model based on approximating the volume of the human body by the volume of a cuboid. Group B is the only group that chose the population density model in task P1, and this group did not change the model in solving task P2 but extended it.

Table 2. Evolution of strategies.

Group	Group Solution of Problem P1	Group Solution of Problem P2
A	Grid distribution (Estimated the number of people that fit in the row and column and multiplied the obtained values.)	Incomplete answer (Measured the width and length of the square, the width of a person and the square area.)
B	Population density	Population density (Proposed removing the unusable area occupied by the statue and fountain from the square area.)
C	Reference point (Divided the quotient of the classroom volume and estimated volume of a person by the ratio between the room height and the estimated height of a person.)	Population density (Proposed removing the unusable area; mentioned using the reference point model.)
D	Grid distribution (Estimated the number of pairs of people that fit through the length and width of the classroom and multiplied the obtained values.)	Population density (Didn't consider unusable area; preformed wrong calculation – divided the square area by the estimated number of people per square meter.)
E	Reference point (Divided the classroom floor area by the estimated area occupied by a person.)	Population density (Didn't consider unusable area.)

As in the original study, this implementation with two Fermi problems shows that the strategies and models change under the influence of the different contexts and tend toward the population density model. The realistic context of the tasks precluded the use of more mathematically sparse problem-solving methods such as exhaustive counting. Interestingly, students did not rely on an external source to solve Task P2. The number of people who were able to occupy Ban Jelačić Square in Zagreb has often been a source of controversy in recent Croatian history. For example, after a demonstration in support of comprehensive curriculum reform on June 1, 2016, police estimated that 25 000 people had gathered in the main square of the Croatian capital, while the organizers' estimate of 40 000 people was much higher. In their Group reports, before solving tasks students first had to give a quick rough estimate of the number of people who might occupy the square. Groups A, C, and D estimated numbers closer to the police estimates (20 000, 17 000, and 7 000, respectively), while Groups B and E were closer to the organizers' estimates (65 000 and 50 000, respectively). Groups B, C and E, which solved the task P2 completely and made no calculation errors, arrived at informed estimates of 31 500, 41 677, and 46 455, respectively.

To emphasize the importance of mathematics and critical thinking in our daily lives, after the group solutions to Task P2 were presented, students were given this information, shown a picture of the protest, and the values calculated by the students were compared to the values calculated by the police and organizers to increase student confidence in their actions and strategies.

7. Discussion

The duration of the activity affected the decision about the number of problems to be solved in the series and the complexity of the realistic context in which the task was set. A more complex everyday context could certainly contribute to some generalization of the population density model, such as noting that there are parts of the area with different population densities. Although students successfully created and applied several different mathematical models within the allotted time frame, I believe that this activity should be extended in some way to allow students more time to conduct independent research and write a more detailed report. One way to accomplish this would be to assign a mathematically equivalent task with a new context for homework or project assignments.

When discussing the solutions obtained, students should be made aware of the assumptions on which the mathematical model was based and how these influenced the definitive answer. Although the organization of work during the classroom activity (switching from individual to group work), along with the construction of the problem, was aimed at encouraging students to question why a particular model better described a realistic situation when selecting a model, there is no evidence that this was achieved. Students were concerned with *how* to determine something, but not with *why* a particular procedure was better for the given problem. The moment they cannot use the procedure because of certain contextual constraints, the current model is discarded, and they switch to a new model. Therefore, if this does not happen spontaneously in the students' work, the teacher should direct the students' focus to the validation step in the modeling process.

8. Conclusion

The didactic potential of set of Fermi tasks used in this study offers several advantages for instruction. Students are confronted with open-ended (realistic) mathematical modeling problems that illustrate the interconnectedness of mathematics with their daily lives and have a stimulating effect on them. It involves a variety of mathematical domains and principles. These include basic arithmetic operations, strong numerical intuition, mastery of units of measurement (e.g. square meters) and the ability to convert between them, the ability to understand geometric shapes, recognize areas of polygons, and decipher spatial relationships. Also includes the ability to make comparisons, and skillfully use proportions and ratios to derive unknown quantities based on known measurements. In addition, it encourages the development of skills such as innovation, creativity, critical thinking, decision making, problem solving, communication, and collaboration.

The advantage of Fermi problems is that they are accessible to students and do not require any special prior mathematical knowledge. Therefore, the modeling activity with a set of Fermi problems can be adapted to students of different educational levels.

Estimating the number of people on a given area, used in this study, exploits the potential of Fermi problems to promote open-ended mathematical modeling and problem-solving skills (Albarracín & Gorgorió, 2018). By solving two versions of the same problem in different contexts, students successfully developed different strategies and models. Although the tasks are open-ended, the sequence of tasks is designed to guide students to develop a particular model. This paper presented a possible example of implementing open-ended mathematical modeling as part of secondary school instruction, along with the strategies and models that could be developed.

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Fermijevi problemi procjena kao aktivnost matematičkog modeliranja u srednjoškolskom obrazovanju

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Sažetak. Matematičko modeliranje iznimno je važan, ali složen proces, te je u prenatrpanim nastavnim planovima i programima ponekad teško naći prostora za njega. Nastavnik ima bitnu ulogu odabira problema i prilagodbe problema svojim učenicima. Prilikom odabira problema modeliranja moguće je odabrati zatvorene tipove problema koji su usmjereni na specifična znanja stoga ih nastavnik može lakše usmjeravati i kontrolirati ili otvorene probleme koji su učenicima kognitivno zahtjevniji. Prednost otvorenih tipova problema modeliranja je da omogućuju učenicima istraživanje prirode promatranog fenomena te kreiranje učinkovite matematičke reprezentacije koja im omogućuje da bolje razumiju stvarnost. Međutim, nastavniku je teško predvidjeti sve moguće interpretacije, strategije i matematičke koncepte koje će učenici primijeniti prilikom stvaranja svog matematičkog modela. Uz to, potrebno je osigurati dovoljno vremena učenicima za istraživanje otvorenog tipa problema, stoga su takvi tipovi problema najrjeđe zastupljeni na nastavi i češće se zadaju kao samostalni ili grupni projekti za domaću zadaću.

U ovom radu bit će prikazano kako grupa sedamnaestogodišnjaka, polaznika srednje škole u Hrvatskoj, rješava dva Fermijeva problema koji zahtijevaju procjenu broja ljudi na zadanoj površini. Osnovna svrha bila je istražiti didaktički potencijal niza Fermijevih problema procjena kao aktivnosti otvorenog matematičkog modeliranja u srednjoškolskom obrazovanju. Karakterizirat će se modeli koje su učenici konstruirali dok su radili individualno i u malim grupama te će se analizirati kako su modeli evoluirali s obzirom na restrikcije nametnute realističnim kontekstom u različitim zadacima.

Ključne riječi: matematičko modeliranje, srednjoškolsko obrazovanje, zadatak otvorenog tipa, Fermijevi problemi procjena, strategije učenika

The Role of Mathematical Picture Books in Teaching the Concept of Zero to First Graders

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Abstract. Over the last two decades, mathematical picture books have increasingly emerged as an innovative teaching approach to support students' understanding of mathematical concepts. The baseline of this paper is a mathematical picture book entitled 70 cherries, which addresses first graders' understanding of the concept of zero. In the theoretical part, the role of this didactic tool for teaching mathematical concepts is presented, and next we focus on the content of our mathematical picture book, which, through an illustrative and imaginative story, introduces the reader to different aspects of the number zero: as the power of the empty set; as a number at the beginning of a positive number line; as a digit in place value notation; as a neutral element in subtraction and addition, and as a symbol to denote the absence of a quantity. In the empirical part, we explore the different functions of the mathematical picture book in teaching the above concept: educational (from the perspective of upbringing and the perspective of learning) and motivational. The research aimed to find out how first grade students recognise the role of the number zero in different contexts, how they accept the picture book as a didactic tool and what the first grade teachers think about the usefulness of the chosen mathematical picture book. The results of the study showed that the use of a mathematical picture book improves pupils' understanding of different aspects of the number zero. The study also founds that teachers recognise the motivational and educational function of this didactic tool, but difficulties are found in the recognition of the educational function due to the lack of teacher knowledge of the different aspects of the number zero. This study presents the first in-depth analysis of the role of the mathematical picture book for mathematics teaching in the Slovenian classroom. It lays the foundations for the implementation and multifunctional usefulness of the proposed new teaching approach for teaching mathematical concepts at the classroom level.

Keywords: picture book, mathematical picture book, zero, didactic tool, teaching mathematics

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Introduction

Many of the concepts we encounter in mathematics have been introduced to us in one form or another before they were formally defined. There is a complex cognitive structure in everyone's mind that generates different personal mental images when a concept is recalled. Usually, in the process of formal introduction, a concept is given a symbol or a name that allows it to be communicated and helps in its mental assimilazion. However, the overall cognitive structure that gives the meaning of a concept is much larger than the retrieval of a single symbol. Tall and Vinner (1981) use the term concept image to describe the entire cognitive structure associated with a concept, which includes all mental pictures and their associated properties and processes. It is built up over years through experiences of all kinds and changes as the individual encounters new stimuli and matures. The representation of a concept and the integration of a variety of external representations are important in building cognitive structures. These allow the child to see the mathematical concept from different perspectives, thus reinforcing their internal representations of the concept and making new connections in their cognitive schema of the concept. The level of understanding is determined by the number and strength of the connections. Connections can be made between different forms of representation of the same idea or between different ideas in the form of the same representation.

In fact, the issue in operating with multiple representations is to generalise from different situations a common essence, an entity that defines a mathematical concept. To be successful in mathematics is to have a rich mental representation of the concept and to be able to switch between external representations. A mental representation is rich if it contains many related aspects of the concept. However, it is only when these are in harmony and when their integration into a single representation takes place that we can speak of abstraction of the concept.

In this paper, we will relate the concept of representation to a mathematical content (the concept of the number 0) and to a didactic tool that facilitates its understanding (a mathematical picture book). In the case of the picture book, we will discuss a relatively new didactic tool that has been increasingly established over the last two decades as an external, graphical representation to support students' understanding of mathematical concepts (Hong, 1996; NCTM, 2000). For the mathematical content, we will focus on the concept of the number zero and the different internal representations of this concept. The latter can cause misunderstandings for students as well as for adults. Our aim was to create a link between these two types of representations, i.e. to show that an external representation in the form of a mathematical picture book can contribute to enriching the internal representation of the number zero.

Picture book

The term picture book is generally defined as a book containing text and illustrations, the two components being linked and forming a relationship with each other. The latter can be simple and illustrative or more complex (Nikolajeva & Scott, 2006), and the importance of their interaction has been emphasised by a number of researchers (Arizpe & Styles, 2003; van den Heuvel-Panhuizen et al., 2014a; Batič & Haramija, 2014; Kobe, 1976; Kobe, 2004; White, 2017). The fundamental components of a picture book are therefore text, illustrations and the relationship between content and form, i.e. interaction (Batič & Haramija, 2014; Haramija & Batič, 2013), which ensure the understanding of the story (van den Heuvel-Panhuizen & Elia, 2012), with the quality of each component playing an important role, as well as their interaction.

A picture book in which the text and illustrations are based on mathematical content is defined as a mathematical picture book (Balić Šimrak et al., 2017). In this case, the interaction between the text and the illustration allows the reader to learn about the mathematical content in a clear, comprehensible and unobtrusive way, while at the same time increasing the motivation to learn.

Many studies show that children's reading of picture books has a positive impact on their mathematical knowledge and the development of mathematical thinking (Cooper et al., 2020; Furner, 2018; Hong, 1996; Hellwig et al., 2000; NCTM, 1989; van den Heuvel-Panhuizen et al., 2014a; van den Heuvel-Panhuizen et al., 2014b; van den Heuvel-Panhuizen et al., 2009; White 2017). Van den Heuvel-Panhuizen et al. (2009) have explored how picture books can support the development of mathematical concepts in young children and how teacher engagement can enhance the power of those features of picture books that support learning. Research has shown that picture books provide a meaningful context for mathematics learning and an informal basis for experiences with mathematical ideas that can be a stepping stone to understanding more abstract mathematical ideas. At the same time, the authors have shown that good quality picture books, not written with the intention of teaching mathematics, have great power to stimulate children to think mathematically. Two studies by van den Heuvel-Panhuizen et al. (2014a and 2014b) examined the impact of reading picture books over time on the mathematical knowledge of children aged 4-6 years. The results showed that the experimental group of children who received a three-month reading programme achieved significantly better mathematical knowledge of number, measurement and geometry than the control group who did not receive this programme.

Research also confirms the positive impact of picture books on children's attitudes towards mathematics, as the use of picture books creates a pleasant, motivating working atmosphere that engages pupils to engage with content that is not necessarily pleasant or simple (Furner, 2018; Hong, 1996; van den Heuvel-Panhuizen et al., 2009; White, 2017). The latter is the reason why children like to reach for picture books even after they have already read them in class (van den Heuvel-Panhuizen et al., 2009). Reading picture books can reduce tension in mathematics lessons and thus improve students' achievement (Furner, 2018). At the same time, the use of picture books in mathematics lessons also allows for good student engagement and participation (van den Heuvel-Panhuizen et al., 2009; van den Heuvel-Panhuizen et al., 2014a; van den Heuvel-Panhuizen et al., 2014b), regardless of how heterogeneous the group is in terms of age, socioeconomic status, mathematical background and language ability (van den Heuvel-Panhuizen et al., 2014a). Van den Heuvel-Panhuizen et al. (2009) also highlight the important role of mathematics picture books in children's language development.

Flevares and Schiff (2014) studied the processes that develop through the use of mathematical picture books and, on this basis, developed a set of reasons for using picture books in mathematics lessons, namely: communication, presentation, integration, problem solving, and reasoning and proving. In order to facilitate the assessment of the quality of mathematics picture books, some experts have developed criteria that specifically emphasise: the relevance and correct exposition of the mathematical content, the possibility of making different connections, the adaptation to the audience (i.e. the child, the learner), and the quality and stimulating nature of the text, illustrations and their interaction (Hellwig et al., 2000: Lipovec, 2017a; Van den Heuvel-Panhuizen & Elia, 2012).

The concept of zero

Understanding the concept of zero is complex and has been a struggle for many cultures in the past. However, there are also difficulties in developing children's mathematical thinking about the concept of zero. The number zero has several roles in a mathematical context: it is the number at the beginning of the number half line; it symbolises the empty place in the place value system; it is a neutral element in subtraction and addition; it represents the difference of two equal values; it represents the power of an empty set; it is a symbol to indicate the absence of a quantity; it is the number that divides positive and negative numbers on a number line.

Much research has been done on children's and adults' understanding of the concept of zero. Hughes (1986), for example, studied the ways children use to represent the absence of a quantity and which are meaningful to children. He put 1, 2, 3 and 0 cubes in 4 tins. After mixing the tins together, the child had to say how many cubes were in each tin. In order to distinguish between tins with different amounts of cubes, the pupils were invited to choose their own ways of indicating the amount of cubes in the tin. It turned out that only one third of the children did not show the correct tin after mixing them again, which means that their choice of quantity labels was meaningful for them and allowed them to identify the specific quantity of cubes. Hughes argues that children do not have significant difficulties in understanding the quantity represented by the symbol 0, but they do have difficulties in understanding the number zero in the place value system.

Catterall (2006) also explored different aspects of understanding the number zero, finding that children often associate the number 0 with the number 1 and therefore place them together when arranging numbers, or associate the number 0 with non-existence or irrelevance and therefore think that it does not matter where the number 0 is placed. In tasks involving computational operations with the number zero, children often ignored the number zero, so that, for example, the result of multiplying by zero was the same as a non-zero number multiplied by zero.

A number of other studies have shown misconceptions associated with the number zero, for example: zero is not a number (Catterall, 2006; Wheeler and Feghali, 1983); zero is not a number, it is just part of the symbol for the number 10 (Evans, 1983); zero is a number, but it develops and exists separately from other numbers and rules (Evans, 1983); zero is valueless/not a number, so it is irrelevant where it lies on the number axis (Catterall, 2006); the result of multiplying by zero is the same as a non-zero number in the calculation (Catteral, 2006), zero enlarges it and when subtracted, it decreases it (Janežič, 2012). At the same time, researchers link the causes of difficulties and misconceptions to several reasons.

Some children are confused by its inconsistent use in society, e.g. the use of the symbol 0 at the beginning of room numbers, telephone numbers, regional numbers, or by inconsistencies in written and oral language; some children confuse the symbol for zero with the letter “O”, and the zero is omitted when naming numbers, e.g. 203 is pronounced ’two hundred and three’ rather than ’two hundred and zero three’, which encourages the idea that zero really means nothingand can be ignored (Russell & Chernoff, 2011). Haylock and Cockburn (2008) link the emergence of misconceptions to the overemphasis on only one of the listed aspects of a mathematical concept, i.e. Catterall (2006) suggests that although children understand the different roles of zero, they may become confused in concrete situations when they do not know which one to consider. Understanding of all roles is built up gradually.

For school children to be offered adequate treatment of the concept of zero, it is necessary that teachers also understand this concept. However, Wheeler and Feghali (1983) found that teachers also have a lack of knowledge about this number. In particular, they have difficulties in understanding some of the computational operations with zero, and some also have difficulties in understanding whether zero is a number at all. A study by Ma (1999, in Russell and Chernoff, 2011) found that teachers in the USA have misconceptions about the number zero, especially within the concept of place value and when splitting the number into decimal units. In the division and multiplication process, some teachers do not understand the division of numbers into different groups of tens and ones, and they also have a poor understanding of place value and the meaning of zero in the place value system.

The concept of zero is a mixture of social evolution, theory in mathematics and procedures. The interplay of all three aspects of understanding zero has an important impact on the development of appropriate, mathematically correct thinking. Interactive, meaningful, authentic learning experiences can lead to a better understanding of zero for both students and teachers (Russell & Chernoff, 2011). Teachers’ and educators’ misconceptions about the number zero point to the need for more attention to developing a more holistic understanding of zero within the curriculum (Wheeler & Feghali, 1983; Russell & Chernoff, 2011).

Problem definition

The mathematical picture book is still a relatively new, unestablished didactic tool for teaching mathematical concepts in the Slovenian classroom. For this reason, there is a lack of research that systematically examines and evaluates this way of teaching mathematics in our environment. To this purpose, in the academic year 2019/2020, the Faculty of Education of the University of Ljubljana, in cooperation with the Mladinska knjiga publishing house, took part in the project On the creative path to knowledge: cross-curricular integration of mathematics and fine arts – a mathematical picture book, within the framework of which, among other things, a mathematical picture book entitled 70 cherries was created. The latter was also the basis for our research. The mathematical aim of this picture book is to introduce children to the concept of zero through concrete life situations, to highlight the different roles of zero, and to show that zero is not a valueless number, but a number whose value is zero.

The 70 cherries picture book is based on the interplay of text and illustrations, which complement each other, and was co-authored by the authors of the text (students of primary teacher education) and the author of the illustrations (a student of fine arts education). Collaboration played an important role in the context of the production, as the authors' different concepts met in the process. It was necessary to reconcile the unity of understanding of the mathematical content with the possibilities of (pictorial) illustration and its full visual impact, understandable for the younger population. At the same time, it was necessary to find situations in the story that are understandable and attractive to children, and that contain mathematical problems in a non-intrusive way.

Description of the picture book

Illustrations: the illustrations had to come up with a range of visual effects that were provided by the story content and to ensure that the right amount of information was paced so that it did not overwhelm the content, but everywhere added visual emphasis, complemented it, or helped to make it more comprehensible. Given the age of the target population, the choice of a vibrant colour palette was deliberate, while at the same time keeping the main protagonists (the numbers) white in order to distinguish them clearly enough from the surrounding. The illustrations are framed to suit the needs of the story, thus focusing the viewer's attention on key information.

Figure 1. The main protagonists of the story
The 70 cherries (Janežič et al., 2022, pp. 2, 4 and 5).

Protagonists: the main protagonists of the story are the numbers 0, 1, 3 and 7 (Figure 1). Each of them has its own character, which is presented in the text and brought to life in the illustrations. **Ena**ja[1] (1) is very competitive and wants to be first in everything. **Tri**na (3) is a bit clumsy but has well-developed empathy. **Sedem**ila (7) is the biggest of the performers, she likes to be in the front, but she is not as competitive as Enaja. **Anič**ka (0) is the gentlest, but always in the background. The young reader learns about numbers through both, names and

[1] The names of the protagonists include the names of numbers in the Slovenian language: ena – one, tri – three, sedem – seven, nič – zero.

images, both of which are strongly reminiscent of real numbers, but at the same time literary and visually compelling enough not to give the impression that it is primarily about mathematics.

Story: In the company of her friends Enaja, Trina and Sedemila, Anička feels insignificant and worthless. They do not wait for her when they go home from the woods. As the smallest number in the column, she is always the last. Her value is zero, which leaves her empty-handed at the cherry stall and she misses the party because she is not allowed to bring any of her friends. In the sports games she is also left without a medal because she fails to jump over the last barrier that she was not allowed to lower. After all this, she no longer wants to hang out with her friends or go to the market. Since it makes sense to follow the dramaturgical form from the point of view of literary persuasiveness for a young pupil, the story ends on a positive note, with not only a mathematical but also an educative point. Anička's plight is noticed by Trina, who is ready to share her portion of cherries with her, and something amazing happens in front of the stall – they get 30 cherries. It is then that Anička realises that she is not worthless at all.

The different scenes in the story highlight the different roles of the number zero.

Scene 1: MARKET: In the market scene, the number zero is mathematically represented in two ways: as the number at the beginning of the number half-bar (Figure 2) and as the power of the empty set (Figure 3).

Figure 2. Number zero as the number at the beginning of the number half line (Janežič et al., 2022, pp. 8, 9).

Translation (p. 8): *Other friends crowded around the stand. "Get in line from biggest to smallest to get the cherries," the hefty saleswoman boomed.*

Translation (p. 9): *Sedemila immediately came to the front. Behind her, Trina was already standing, and in a burst of joy, she kicked the crate and knocked it over. Enaya was grumbling loudly the whole time. She had to wait for the mess to be cleaned up. But even more than that, she was annoyed that for once she was not the first.*

Figure 3. Number zero as the power of the empty multitude (Janežič et al., 2022, pp. 10–11)

Translation (p. 10): *The last in the queue waited patiently for her cherries Anička. The saleswoman gave Sedemila seven of them, Trina three, and Enaja just one.*

Translation (p. 11): *Finally, it was Anička's turn to receive the cherries. A sign caught her eye. She slowly read: "AS MUCH AS YOU ARE VALID, AS MANY CHERRIES YOU CAN TAKE."*

Scene 2: BIRTHDAY: In the birthday party scene, the number zero is also emphasised as the power of the empty set: Anička receives an invitation to Sedemilla's birthday party, but she asks the invitees to bring as many friends as they are worth. Anička was frustrated therefore she decided not to attend the party.

Scene 3: SPORT GAMES: This situation shows zero as a neutral element for subtraction (Figure 4).

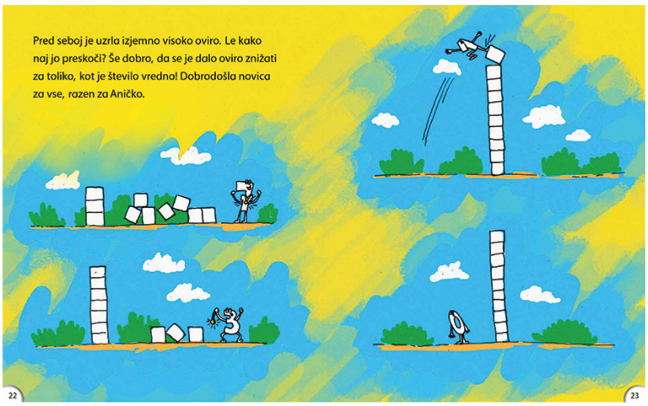

Figure 4. Number zero as a neutral element in subtraction (Janežič et al., 2022, pp. 22–23)

Translation (p. 22): *She saw an extremely high barrier in front of her. But how should she jump over it? It was a good thing that the barrier could be lowerer as much as the number is worth! Welcome news for everyone, except for Anička.*

Scene 4: EPILOGUE: In the last scene of the revisit to the market with Trina, zero is presented as a digit in the place value notation (Figure 5).

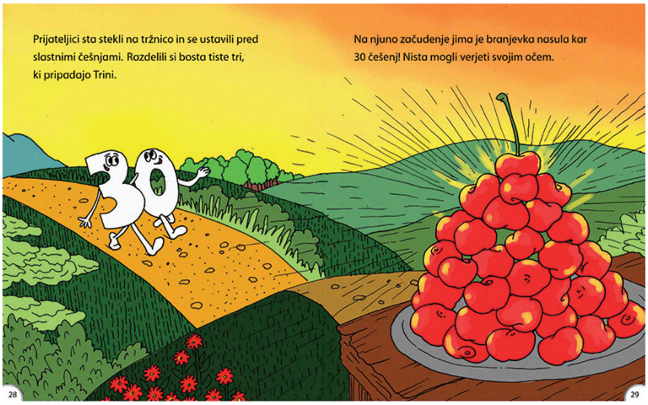

Figure 5. Zero as a digit in the place value notation of a number (Janežič et al., 2022, pp. 28–29)

Translation (p. 28): *The two friends ran to the market and stopped in front of delicious cherries. They will share the three, that belong to Trina.*	Translation (p. 29): *To their amazement, the saleswoman gave them 30 cherries! They couldn't believe their eyes.*

Survey

The aim of the study was to find out how the mathematical picture book 70 cherries influences the understanding of the concept of zero by 1st grade pupils, how it is perceived as a didactic tool and what the 1st grade teachers think about the usefulness of the selected mathematical picture book. The aim was to check whether the mathematical picture book achieves in practice the objectives for which it was created.

Research questions

1. Does using the 70 cherries mathematical picture book improve students' understanding of the number zero in different contexts?
2. Do 1st grade pupils recognise the educative aspect of the 70 cherries mathematical picture book?
3. Do teachers recognise the learning, motivational and educative function of the 70 cherries mathematical picture book?

Method

We used descriptive and causal experimental methods. The research was divided into two parts. In the first part, we examined the students' achievements. We used a combination of a quantitative approach (pre-test before the mathematics picture book lesson and post-test after the lesson) and a qualitative approach (lesson observation). In the second part, we checked teachers' views using a quantitative approach (questionnaire survey).

Sample

The sample was a non-randomised casual sample. It included 84 pupils in five 1st grade classes from three primary schools and 18 first triad teachers from six Slovenian primary schools.

Instrument

Three data collection instruments were used: a knowledge test, an observation sheet, and a questionnaire. The first part of the study had three steps: an initial check of understanding of the number zero by means of a knowledge test, a pedagogical experiment (teaching with a mathematical picture book) and a final check of understanding of the number zero by writing the same knowledge test again. The test consisted of 8 mathematical tasks related to different concepts of zero. For the observation, we used an observation sheet divided into six sections (educational aspect, motivational aspect, educational aspect, pupils' reactions, feelings, and attitudes towards the subject and other observations), and for each section we recorded the observations, the way the data were collected and specific descriptions of the children's reactions. We ensured equivalent conditions in the different classes by preparing a detailed lesson preparation in advance, which was presented in detail to the class teachers before the implementation and followed by them during the lesson. In addition to recording data using an observation sheet, all lessons were audio-recorded. A few days after the lesson, we re-ran the knowledge test under the same conditions as the first time. In the second part of the study, we examined teachers' attitudes towards teaching with mathematical picture books. We used an online questionnaire for teachers in the first triad to find out how familiar they were with the concept of mathematical picture books as the didactic tool and what their opinions were on the 70 cherries mathematical picture book. Before filling in the questionnaire, we also provided them with a link to an online list server where they could view the picture book.

Data processing

The data obtained from the questionnaire were analysed using descriptive statistics (frequencies and arithmetic means), and the comparison of students' performance on the pre- and post-tests was determined using inferential statistics (Wilcoxon test). The lessons were analysed based on observation sheets. The data obtained were grouped into meaningful classes, analysed according to the categories set and supplemented with examples of narrative dialogues between the teacher and the pupils.

Results

Improving understanding of the number zero in different contexts

First, we wanted to find out whether the mathematical picture book 70 cherries contributes to a better understanding of the number zero in different contexts. With the pre- and post-test problems (Appendix 1) we tested 6 different roles of zero, i.e.: zero as the power of an empty set, zero as the number at the beginning of a number half-line (or as the number that is smaller than all natural numbers), zero as a neutral element in subtraction/addition, zero as the difference of two equal numbers, zero as a digit in place value notation and zero as a number with the value zero. Table 1 shows the pupils' performance on each task and role of number zero.

Table 1. Pre-test and post-test success performance

Task	**Role of number zero**	**Pre-test (%)**	**Post-test (%)**	**Difference in increase (%)**
1	An empty set	93	100	7
2	A number with the value zero	61	86	25
3	The beginning of a number half-line	70	89	19
4a	The beginning of a number half-line	86	94	8
4b	a digit in place value notation	67	86	19
5	An empty set	88	99	11
6	A difference of two equal numbers	70	87	17
7	A neutral element in addition/ subtraction	57	79	22
8	A neutral element in addition/subtraction	51	69	18

Pupils improved their performance on all types of tasks on the post-test. The tasks related to the empty set aspect were solved best in the pre-test, so there was not much room for improvement in the initial performance and consequently the progress in knowledge of this type of tasks is smaller. However, the greatest progress was made in recognising zero as a number with a value of zero rather than as something without a value.

We were interested in whether the differences in pupils' pre- and post-test performance were statistically significant. Due to the non-normal distribution of the data (Kolmogorov-Smirnov test $\alpha = 0.000$), we used a non-parametric test, the Wilcoxon test. The results confirmed that there are statistically significant differences between the mean scores of pupils on the pre-test and the post-test ($Z = -5.642$; $\alpha = 0.000$). The data can be generalised to the baseline set and, with a risk of less than 0.1 %, it can be argued that, even in the baseline set, pupils on the post-test ($M = 11.69$) would have a higher mean score than on the pre-test ($M = 9.90$).

The identification of different aspects of zero in different contexts is also confirmed by the analysis of the observation sheets. We summarise the key observations after reading the picture book and discussing it with the pupils.

Zero as the power o fan empty set	The pupils noticed that one of the illustrations was without Anička's hands, but if they had been drawn, they would have been empty, without cherries. They explained that she was not allowed to bring anyone to the party because she was zero, however, she could have gone to the party on her own, but she did not. They understood that Anička was worth zero and that it was less than one, three or seven.
Zero as the beggining of the half-line	The pupils were able to list the order of the main protagonists in the column in front of the sales stand. They knew that the numbers had to be placed in a column from the largest to the smallest, Sedemila is first not because she is the tallest in appearance but because she is the largest number, and Anička is the last because she is the smallest number and is worth zero.
Zero as a digit in a place-value notation	After the question with which the story ends, the pupils were able to answer that Anička and Sedemila will get seventy cherries. When asked why Trina and Anička ended up with thirty cherries, they answered that when they get together, they get thirty cherries. They had no problem answering all the possible combinations of how many cherries a certain number and Anička would get, and that they would get the most cherries if they all went to the market together. But if Anička invited more zeros together with Enaja, they started to list that three zeros would get one thousand and four zeros would get ten thousand cherries.
Zero as a neutral element for subtraction	In the Sports Games scene, the pupils understood that this task was the hardest for Anička and the easiest for Sedemila. When asked what would happen if the friends had to raise the barrier by what they were worth, they answered that Anička's barrier would be the lowest because it would not go up, and Sedemila's barrier would be the worst because she would have to raise the barrier by the biggest number of cubes, i.e. seven. They thought that Anička would then be lucky because she would have the lowest hurdle and would be the best.

Identifying the educative aspect of the 70 cherries math picture book among pupils

The identification of the educative aspect of the mathematical picture book by the pupils was determined through observation and transcription of conversations between the teacher and the pupils. Here is an example of a guided conversation:

T: "How did our Anička feel in this story?"

A: "Sad."

T: "She felt sad, didn't she? Did you understand why was she sad? Why was she so unhappy?"

B: "Because she didn't get any cherries and because she didn't win, because she was nothing."

T: "Is that quite true?"

ALL: "No!"

T: "Is Anička really worthless?"

ALL: "No!"
T: "So what is she?"
C: "She is worthy. In the end, they get thirty cherries together."
T: "Can we all learn something from this too?"
ALL: "Yees!"
D: "Yes, we are all worth too."

Other messages that the pupils recognised in the story were:

— *"... that we are all worth something/that everyone is worth something/even if you think you are worthless, you are worth something."*

— *"... that you are kind."*

— *"... that it's nice to share things, just like Trina and Anička did."*

— *"... if someone doesn't like you, there is also someone that likes you too, not just dislike you/that your friends help you."*

We find that, with the teacher's guidance, the pupils recognise several educative messages in the story, namely that everyone is worth something, that it is valuable to have a friend, to share things with them, to help each other in times of need and to be kind.

Teachers' identification of the learning, motivational and educative function of the mathematical picture book

The second part of the study was aimed at identifying teachers' attitudes towards the use of the mathematical picture book 70 cherries in mathematics lessons and identifying its learning, motivational and educative functions. In the following, we analyse the answers to the three questions of the online questionnaire.

1. Teachers were asked to select their level of agreement with the statement on a 5-point Likert scale (Table 2).

Table 2. Teachers' opinions on the usefulness of the 70 cherries math picture book.

Statement	**Answers**					**Mean[2]**
	1	**2**	**3**	**4**	**5**	
	Strongly agree	**Agree**	**Neither agree nor disagree**	**Disagree**	**Strongly disagree**	
Maths is meaningfully integrated into the story.	8 (44 %)	9 (50 %)	1 (6 %)	0 (0 %)	0 (0 %)	1,6
The 70 cherries maths picture book is a useful tool to understand the concept of the number 0.	7 (39 %)	10 (56 %)	1 (6 %)	0 (0 %)	0 (0 %)	1,7
Working with a selected maths picture book allows pupils to be actively involved.	5 (28 %)	6 (33 %)	4 (22 %)	3 (17 %)	0 (0 %)	2,3

[2] Strongly agree. = 1 p, Agree. = 2 p, Neither agree nor disagree. = 3 p, Disagree. = 4 p, Strongly disagree. = 5 p

The 70 cherries maths picture book offers maths challenges for pupils.	6 (33 %)	8 (44 %)	3 (17 %)	1 (6 %)	0 (0 %)	1,9
The story in the 70 cherries math picture book would motivate pupils.	9 (50 %)	8 (44 %)	1 (6 %)	0 (0 %)	0 (0 %)	1,6
The text in the picture book 70 cherries is appropriate for the developmental level of the children.	7 (39 %)	10 (56 %)	0 (0 %)	1 (6 %)	0 (0 %)	1,7
The illustrations in the 70 cherries picture book are appropriate for the developmental level of the children.	9 (50 %)	8 (44 %)	1 (6 %)	0 (0 %)	0 (0 %)	1,6
The illustration in the 70 cherries math picture book would motivate pupils.	10 (56 %)	6 (33 %)	2 (11 %)	0 (0 %)	0 (0 %)	1,6
Illustrations and text are appropriately complemented and connected.	11 (61 %)	7 (39 %)	0 (0 %)	0 (0 %)	0 (0 %)	1,4
The 70 cherries maths picture book has an educative impact.	10 (56 %)	6 (33 %)	1 (6 %)	1 (6 %)	0 (0 %)	1,6
The 70 cherries maths picture book could be useful also at some other school subjects.	4 (22 %)	13 (72 %)	1 (6 %)	0 (0 %)	0 (0 %)	1,8
The teacher's role in identifying all the aspects of the number zero included in the picture book is crucial.	5 (28 %)	9 (50 %)	3 (17 %)	1 (6 %)	0 (0 %)	2,0
Parents could do the same quality of reading and presenting the content of the picture book as teachers.	2 (11 %)	6 (33 %)	7 (39 %)	3 (17 %)	0 (0 %)	2,6
I would recommend the 70 cherries picture book to other teachers for use in maths lessons.	5 (28 %)	10 (56 %)	2 (11 %)	1 (6 %)	0 (0 %)	1,9
I would recommend the 70 cherries picture book to other teachers for use in their lessons to educate.	3 (17 %)	12 (67 %)	3 (17 %)	0 (0 %)	0 (0 %)	2,0
Maths picture book 70 cherries could enrich maths lessons.	9 (50 %)	7 (39 %)	2 (11 %)	0 (0 %)	0 (0 %)	1,6

Teachers convincingly agree that the 70 cherries mathematical picture book is a useful tool for understanding the concept of zero (agree, 56 %; strongly agree, 39 %). They also agree (56 %) or strongly agree (39 %) that the text in the 70 cherries picture book is appropriate for the children's developmental level, while they find the illustrations even more appropriate. The prevailing opinion is that working with the chosen mathematical picture book allows for active involvement of the pupils. The role of the teacher in identifying all aspects of the number zero included in the picture book is not seen as crucial by all (17 % neither agree nor disagree). Teachers are divided on whether parents could provide the same quality of reading and presentation of the content of the picture book as teachers. Most teachers (39 %) are undecided, but more of those who are undecided agree than disagree. At the same time, the majority also agree that a mathematical picture book 70 cherries could enrich mathematics lessons (strongly agree, 50 %; agree, 39 %).

2. Teachers were asked to choose from the given options those mathematical situations related to the concept of zero that they identified in the story (Figure 6).

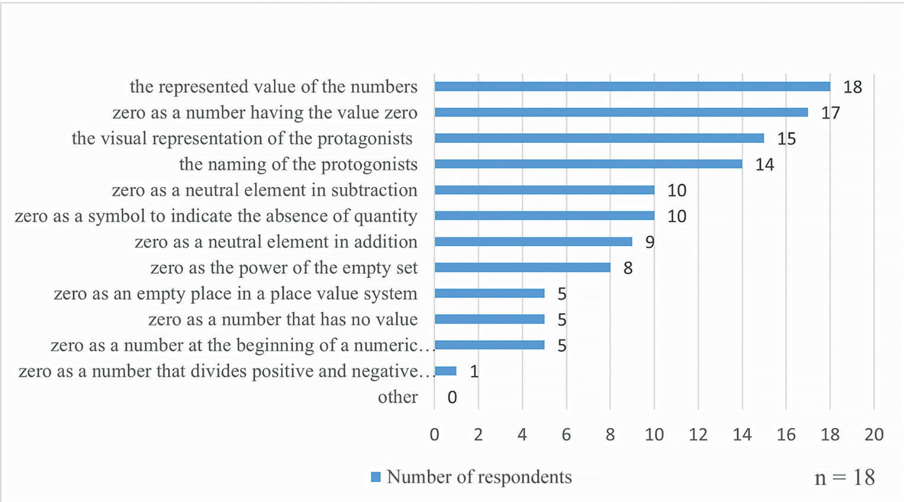

Figure 6. Recognition of different mathematical situations in the picture book 70 cherries.

The results show that teachers recognise different mathematical situations in the picture book, but most teachers do not recognise all of them. The vast majority recognise zero as a number having the value zero (94 %), the visual representation of the protagonists in the form of numbers (83 %) and the names of the numbers included in the naming of the protagonists (78 %). The other situations are only recognised by a good half of the teachers (55 %) or less. We were surprised to find that the concept of the number zero as an empty place in the place value system was recognised only by 28 % of the teachers surveyed, even though this very concept is strongly emphasised in the conclusion of the picture book. There are also some misconceptions: in the story, some teachers identify the number zero as having no value, even though this is the exact misconception that the picture book is trying to discourage or negate. Some teachers referred to both the concept of zero as having a value of zero and the concept that zero as a number that has no value, which is contradictory and again points to the teachers' misconceptions about the number zero.

3. Several possible uses of the picture book in the classroom were listed and teachers were asked to rank them in order of importance from 1 (least important) to 6 (most important) (Table 3).

Table 3. Assessing the importance of the purposes of using picture books in the classroom.

	The purpose	No. of respondents for each importance rating (1 = least, 6 = most)						M	SD
		6	5	4	3	2	1		
1.	As a motivational tool	6 (35 %)	2 (12 %)	3 (18 %)	3 (18 %)	2 (12 %)	1 (6 %)	4,2	1,68
2.	To introduce a new maths concept through a story	3 (18 %)	6 (35 %)	2 (12 %)	4 (24 %)	2 (12 %)	1 (6 %)	4,0	1,54
3.	To assess understanding of a mathematical concept	3 (18 %)	3 (18 %)	4 (24 %)	1 (6 %)	3 (18 %)	3 (18 %)	3,6	1,80
4.	To stimulate discussion between teacher and pupils or between pupils themselves	1 (6 %)	2 (12 %)	4 (24 %)	6 (35 %)	4 (24 %)	0 (0 %)	3,4	1,18
5.	As an educative tool	0 (0 %)	4 (24 %)	3 (18 %)	5 (29 %)	5 (29 %)	0 (0 %)	3,4	1,17
6.	I wouldn't use it in lessons	3 (18 %)	0 (0 %)	1 (6 %)	1 (6 %)	1 (6 %)	11 (65 %)	2,2	1,99

Teachers most often chose the most important purpose of using picture books in the classroom as a motivational tool (35 %), which also has the highest average ($M = 4.2$), they also identify an learning function (statements 2 and 3), and to a lesser extent an educative function ($M = 3.4$). However, the data are very scattered and it is difficult to draw uniform conclusions from them.

Discussion

The study shows that the 70 cherries picture book improves mathematical knowledge or pupils' understanding of the number zero in different contexts. The findings are consistent with comparable studies, which also confirm the positive impact of mathematics picture book instruction on overall performance in the subject or on the development of mathematical thinking (Hellwig et al., 2000; NCTM, 1989; van den Heuvel-Panhuizen et al., 2014a; van den Heuvel-Panhuizen et al., 2014b; van den Heuvel-Panhuizen et al., 2009; White 2017).

In general, mathematical picture books offer opportunities for mathematical discourse, support mathematical thinking through illustrations and can be used as a starting point for various activities, especially in relation to problem solving (Flevares and Schiff, 2014). For the 70 cherries mathematical picture book, teachers recognise the usefulness in understanding the mathematical concept under discussion (95 %), but are slightly less convinced that it would offer mathematical challenges for pupils (77 %) or stimulate their activity (61 %).

According to Zupančič (2012), illustrations are the first motivational means by which children decide whether a picture book will be interesting enough to read and they also motivate the reader during the reading process by looking for somewhat hidden objects, persons and messages, or by anticipating what will happen next in the story. According to Fang (1996), the illustrations continuously maintain a slight tension between the depiction of events and the flow of the story. According to the teachers, the 70 cherries picture book has motivational potential. They attribute this potential in particular to the illustration, which they consider to be complementary and appropriately linked to the content of the story. It can therefore be concluded that teachers recognise a motivational function in the 70 cherries mathematical picture book.

A good quality picture book text should have an educative and/or cognitive message (Haramija & Batič, 2013). The 70 cherries picture book also contains strong educative messages. The story highlights in particular the problem of worthlessness. It conveys the message that everyone is worth something, even if it sometimes does not seem so at first sight, and invites the reader to be able to recognise the distress of close ones and to be supportive and helpful when needed. However, each reader can also extract other educative messages that he or she recognises in the story. The results confirm that the educative effect unobtrusively embedded in the story is recognised by both pupils and teachers. The majority of respondents would recommend the picture book to other teachers for educative purposes. In the classroom, they would use the mathematical picture book 70 cherries as an educative tool, but this would not be the most important purpose.

However, difficulties arise in identifying the learning function. Although teachers attach great importance to the learning purpose of the mathematical picture book 70 cherries, misconceptions occur in understanding the mathematical background of the story, i.e. in understanding the different concepts of zero that appear in the story. A small proportion of teachers recognised the number zero as the power of the empty set, the number zero as the empty place in the place value system and the number zero as the number at the beginning of the number half-line. A few teachers identified zero as a number that has no value, indicating that teachers have misconceptions about zero and that there is also a misunderstanding of the mathematical content of the picture book or its message. We reached the same conclusions as other studies on teachers' perceptions and understanding of zero (Russell & Chernoff, 2011; Wheeler and Feghali, 1983), which found that teachers' knowledge of the number was deficient.

Certainly, among the results, it is worth highlighting the aspect of identifying zero as a number with a value (86 % of the children in our study). Indeed, it is a very abstract concept that even adults have difficulties with (28 % of the teachers in our study identified zero as a number that has no value). None of the scenes in the story directly refer to this aspect, but it is present throughout the experience of the story and of the protagonist, Anička, who in the epilogue turns out to have a value like other numbers. It could hardly be said that the children's answers reflect their understanding of the mathematical property of the number zero, but rather their identification with Anička, who until the last scene feels worthless – that is, without value. However, the last scene refers to the context of increasing the number by adding the digit zero to the end of the number notation – i.e. the place value system, not to the abstract concept of zero as a number with a value of zero. In this case, we believe that there has been a transfer of understanding of the phrase "to have value" from a real-life situation to the mathematical world, where it has a more abstract meaning. It is therefore appropriate to ask what is the key difference between the statements "zero is a number with a value of zero" and "zero has no value". Again, as with other more complex mathematical concepts (Hughes, 1986; Kilhamn, 2011; Sfard, 1991), the development of the concept of zero parallels the development of the concept in the individual and the development of the concept throughout human history. The analysis of the difficulties faced by the individual at a given moment mirrors the full spectrum of difficulties faced by ancient civilizations and experts up to modern times: from the perception of zero as an emptiness to zero as a number equivalent to all other numbers (Seife, 2000; Evans, 1983). The statement "zero has no value" is not synonymous with describing the absence of quantity; zero has a value that is attributed to the set with the absence of quantity, i.e. the empty set. Following Tall and Vinner (1981), it can be concluded that the story of the picture book has contributed to broadening the concept image of the notion of zero, but it is difficult to draw a clear line between understanding the different aspects of zero and identifying the reasons that affect the success of solving zero number problems. We can suspect that in some cases the children's intuitive ideas, linked to the story, rather than a deeper understanding of the mathematical abstract concept, are behind the correct answers. The story of the mathematical picture book 70 cherries can be seen as a starting point that has established a good foundation in the child's perception of the number zero, on which the concept of zero will need to be built upon and developed in further education.

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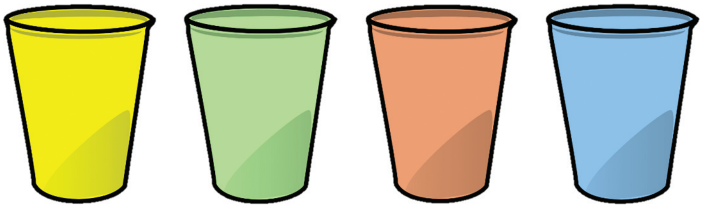
Apprendix 1: Knowledge test

1. THERE ARE FOUR POTS IN FRONT OF YOU. INDICATE HOW MANY CUBES ARE IN EACH CUP.[3]



2. IS ZERO A NUMBER? CIRCLE.

YES **NO**

3. ORDER BY SIZE FROM SMALLEST TO LARGEST.[4]

4. CIRCLE THE LARGEST NUMBER IN GREEN.
CIRCLE THE SMALLEST NUMBER IN BLUE.

100 **10** **1000000**

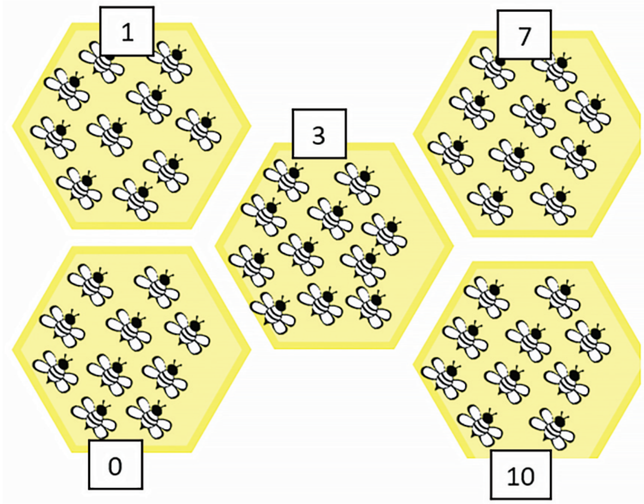
1

9 **1000** **0**

[3] 4 cups are placed in front of the pupils in the same order as they are drawn on the paper. In each cup we put a certain number of cubes (yellow: 2, green: 3, red: 0 and blue: 1 cube). The pupils have to mark how many cubes are in each cup in away they wish.

[4] Each pupil is given stickers with numbers written on them (0, 1, 3, 5, 7). They have to stick the stickers in the corresponding green square according to the instruction from left to.

5. COLOUR THE APPROPRIATE NUMBER OF BEES.



6. GRANDMA BAKED 5 MUFFINS FOR HER GRANDCHILDREN.
2 WERE EATEN BY ŠPELA AND 3 BY MATEJ.
HOW MANY MUFFINS DID SHE HAVE LEFT? ______________

7. ŠPELA WAS HOLDING 5 BALLOONS. A STRONG WIND BLEW. NONE OF THE BALLOONS BLEW AWAY. COLOUR THE BALLOONS THAT ARE LEFT.

8. CAKE COSTS 4 €. ŠPELA HAS 3 € AND MATEJ 0 €. DO THEY HAVE ENOUGH MONEY TO BUY A CAKE TOGETHER?

ANSWER:

YES HOW MUCH MONEY DO THEY HAVE LEFT? __________

NO HOW MUCH MONEY DO THEY HAVE MISSING? __________

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Vloga matematičnih slikanica pri poučevanju koncepta števila nič pri prvošolcih

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Povzetek. V zadnjih dveh desetletjih se matematične slikanice vse bolj uveljavljajo kot inovativen učni pristop, ki učencem pomaga pri razumevanju matematičnih pojmov. Izhodišče tega prispevka je matematična slikanica z naslovom 70 češenj, ki obravnava razumevanje koncepta števila nič pri prvošolcih. V teoretičnem delu je predstavljena vloga tega didaktičnega orodja pri poučevanju matematičnih pojmov, nato pa se osredotočimo na vsebino naše matematične slikanice, ki skozi nazorno in domiselno zgodbo bralcu predstavi različne vidike števila nič: kot moč prazne množice; kot število na začetku pozitivne številske osi; kot števko v zapisu mestne vrednosti; kot nevtralni element pri odštevanju in seštevanju ter kot simbol za označevanje odsotnosti količine. V empiričnem delu raziščemo različne funkcije matematične slikanice pri poučevanju omenjenega pojma: izobraževalno (z vidika vzgoje in z vidika učenja) in motivacijsko. Z raziskavo smo želeli ugotoviti, kako prvošolci prepoznavajo vlogo števila nič v različnih kontekstih, kako sprejemajo slikanico kot didaktični pripomoček in kaj o uporabnosti izbrane matematične slikanice menijo prvošolski učitelji. Rezultati raziskave so pokazali, da uporaba matematične slikanice izboljša razumevanje različnih vidikov števila nič pri učencih. Raziskava je tudi pokazala, da učitelji prepoznavajo motivacijsko in izobraževalno funkcijo tega didaktičnega pripomočka, vendar se pri prepoznavanju izobraževalne funkcije pojavljajo težave zaradi pomanjkanja znanja učiteljev o različnih vidikih števila nič. Študija predstavlja prvo poglobljeno analizo vloge matematične slikanice pri pouku matematike v slovenskem razredu. Postavlja temelje za izvajanje in večnamensko uporabnost predlaganega novega učnega pristopa za poučevanje matematičnih pojmov na razredni stopnji.

Ključne besede: slikanica, matematična slikanica, število nič, didaktični pripomoček, poučevanje matematike

Word Problems in the Textbooks for Primary Mathematics Education in Croatia

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Abstract. Word problems are an indispensable part of mathematics education. In the literature, they are tasks set up in real-life, imaginary or mathematical contexts solved using arithmetic operations on numerical data. Word problems can be classified using different criteria: the type of context, arithmetic and semantic structures, and the function of graphics accompanying the task. We performed a content analysis of selected textbook units covering all arithmetic operations across six textbook editions and all four grades of primary education. The results are hence informative about the implemented methodology and indicative of the distribution of different types of tasks. The classification proposed in this paper is exhaustive, unambiguous and appropriate for the analysis of word problems in primary mathematics textbooks regardless of grades and editions. Results of the study suggest that particular types of contextual problems are more common than others and that problems that have an arithmetic structure with operand unknown or multi-step problems are more commonly mathematical than contextual. Providing diverse opportunities in working with contextual problems is relevant knowledge for teachers, textbook authors, scientists and policymakers hence further research following the proposed methodology should be undertaken.

Keywords: word problem, contextual problem, graphic representation, primary mathematics education, mathematics textbook, content analysis

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1. Introduction

Since the early civilisations word problems have been mediators for teaching mathematics. The Rhind Papyrus found in ancient Egypt claims for itself to be "a thorough study of all things, insight into all that exists, knowledge of all obscure secrets" whereas it includes a range of arithmetic and practical problems, for example:

> *"In each of seven houses, there are seven cats; each cat kills seven mice; each mouse would have eaten seven sheaves of wheat; and each sheaf of wheat was capable of yielding seven hekat measures of grain. How much grain was thereby saved?" (Burton, 2011).*

To this day, word problems in school mathematics are used to connect mathematical content with daily, real-life situations. Students' skills in solving real-life problems are directly affected by the variety of word problems and situations they encounter (Verschaffel et al., 2020). For that reason, we examined the type of word problems presented in mathematics textbooks for primary education in Croatia.

2. Word Problems in Mathematics Education

There are different interpretations of the term word problems. Verschaffel et al. (2010, 2020) restricted them to tasks set up in real-life or imaginary contexts and solved using arithmetic operations on numerical data. An example of such a problem is the following:

> *Example 1. A group visit to the planetarium costs* 3 € *per person. How much money will the group of 15 classmates and their teacher pay in total for their visit?*

There are two types of problems given in written form that do not fall into the description above. Mathematical arithmetic tasks use purely mathematical context and formal terms for operands and operations. An example of such a problem is the following:

> *Example 2. Find the other factor if one factor is 16 and the product is 2000.*

Problem tasks are set in an authentic context and require heuristics in the solution process (Verschaffel et al., 2020). An example of such a problem would be:

> *Example 3. We [a class] are planning a trip to the city to visit the planetarium. Organise the trip and the visit with minimum cost.*

Word problems have different roles in teaching mathematics: *motivation* when using relatable context in tasks to raise interest in studying mathematics, *concretisation* when using familiar context to represent a new abstract concept, *application* in the sense of using them for practising the learned procedure, and *modelling* in the sense of engaging students in problem-solving heuristics. For the latter cause, word problems students encounter need to be ambiguous, non-trivial and simulate authentic situations. Consequently, the application role conflicts with the modelling role since it assumes routinely applying an arithmetic operation when solving a set of word problems. A particular issue is the implicit connection of certain verbal cues and expressions with the specific arithmetic operation (Daroczy et al., 2015; Polotskaia & Savard, 2021; Verschaffel et al., 2020). For example, in a word problem

Example 4. Anna has 5 coins. Hannah has 2 coins more than Anna. How many coins do they have altogether?

the student interprets only the relationship 'have altogether" and without solving the intermediate step calculates the sum of given numbers; or in a word problem

Example 5. Hannah has 7 coins. She has 2 coins more than Anna. How many coins does Anna have?

the student notices the relationship 'more than" and interprets it routinely as the sum of given numbers instead of solving for the unknown summand.

2.1. Classification of Word Problems

Different features of word problems affect their difficulty and can be used for classification. The arithmetic structure depends on the number of operations required, whether the problem is one step or multi-step problem; the unknown value, whether the result or operand is unknown; and the solution process, whether basic calculations for one-step or multi-step problems, equations or heuristics are required (Daroczy et al., 2015; Verschaffel et al., 2020). Multi-step problems are more difficult than one-step problems.

Assuming word problems are tasks given in the written form, they can be distinguished as mathematical, contextual and authentic problems following the descriptions given above. However, some researchers suggested that contextual word problems can be standard or problematic (Polotskaia & Savard, 2021; Savard & Polotskaia, 2017; Verschaffel et al., 2010, 2020). The standard problems are formulated as typical word problems and solved with simple calculations. Problematic tasks are formulated to resemble standard problems but some hidden or indistinct assumptions hinder using simple arithmetic in the solution process. An example of such a problem would be:

Example 6. Mary had 30 books. She wants to arrange them on 3 shelves. How many books will there be on each shelf?

where a student shouldn't assume that all books and shelves are of the same width and length. Emphasising the application role of word problems contributes to deepening the misinterpretation of problematic tasks. Students routinely solve the problem using seemingly most appropriate calculations without considering the contextual realm. When faced with problematic tasks in a non-scholastic environment or with standard word problems before instruction students approach them heuristically (Polotskaia & Savard, 2021; Savard & Polotskaia, 2017; Verschaffel et al., 2010; Verschaffel & De Corte, 1993). A contextual word problem needs to be realistic when representing aspects and conditions of the out-of-school problem situation.

2.1.1. Semantic Structure of Word Problems

Semantic structure relates to the contextual relationship between values in the problem. One-step problems can be additive or subtractive, multiplicative or division problems and they can be further classified. The classes of additive problems are following (Savard & Polotskaia, 2017; Singh et al., 2020; Van de Walle et al., 2016; Verschaffel et al., 2020; Verschaffel & De Corte, 1993; Vicente et al., 2022):

— *Change* given an initial set there is a dynamic increasing or decreasing change that affects the result set,

— *Combine* there is a superset that is joined from or separated into two subsets,

— *Compare* compared value is that much more/less than the reference value.

Another category named *Equalise* is sometimes suggested but these problems can be categorised as change problems to be realised in the future (Elia, 2020; Vicente et al., 2022). Problems in each class can be expressed as additive or subtractive and with different values as unknown. Corresponding changes affect the arithmetic structure of the word problem. The classification cannot be generalized to multi-step problems because various combinations can be used, e.g. problem in Example 4 includes *Compare* and *Combine* relationships. One-step problems with the same arithmetic structure and different semantic structures have different difficulties (Daroczy et al., 2015; Elia, 2020; Verschaffel et al., 2020; Verschaffel & De Corte, 1993). When comparing for the same arithmetic structure, semantic structure ordered by difficulty are *Change*, *Combine* then *Compare*. *Compare* problems with the difference unknown are more difficult in subtraction than addition. Within the same semantic structure, problems are more difficult if operands are unknown, as in Example 5, than if the result is unknown, as in the following example

Example 7. Hanna has 7 marbles. Anna has 2 marbles less than Hanna. How many marbles has Anna got?

The classes of multiplicative word problems are following (Polotskaia & Savard, 2021; Van de Walle et al., 2016; Verschaffel et al., 2020; Vicente et al., 2022):

— *(Equal) Groups* the total number of items depends on the number of equipotent groups and the number of items in each group,

— *Rate* total value depends on the number of unit items and the value of the unit item,

— *Compare* compared value is that many times more/less than the reference value,

— *Array* total number of items depends on the number of items per row and per column,

— *Cartesian product* total items are a combination of two items, each from a different set.

Polotskaia and Savard (2021) differed multiplicative structures as multiplicative composition, comparison and (Cartesian) product. Referring to the classes above, composition structure includes *Rate*, *Groups* and *Array* problems where the whole is multiple of the unit (numeric or measured) value of the same type, comparison structure coincides with *Compare* problems and the product structure covers multiplicative problems where the resulting value is of a different type than initial values, for example calculating the area of a rectangle. Multiplicative structures are not grounded solely on consecutive addition (Savard & Polotskaia, 2017). Depending on the roles of operands in different additive and multiplicative classes of semantic structures problems can be commutative – additive *Combine*, multiplicative *Array* and *Cartesian product*, or non-commutative (Verschaffel et al., 2020). Division problems can be arranged in the same classes of semantic structures as multiplicative problems. They differ in arithmetic structure as a partitive division when the quotient is unknown, and a qoutitive division when the divisor is unknown (Van de Walle et al., 2016; Verschaffel et al., 2010). Division word problems can be problems with or without remainder, and the remainder can be interpreted as a leftover, fraction, next closest whole number or approximation.

2.1.2. Representation of Word Problems

The solution to word problems can be symbolic (numbers and signs), graphic or a combination of both. Graphics accompanying word problems have different functions

— *Decorative* – when graphics illustrate the theme of the context without information on values or relations;
— *Informational* – when graphics provide (some or all) object-related information necessary to solve the problem,
— *Organisational* – when graphics provide visual or diagrammatic information about relations between values in the problem that directly affects the solution process (Boonen et al., 2014; Elia, 2020; Ott, 2020; Verschaffel et al., 2020; Vicente et al., 2022).

Graphic representations, in particular organisational graphics, are useful when they accurately represent a relationship, especially in non-routine problems (Boonen et al., 2014; Ott, 2020; Verschaffel et al., 2020). Students' performance was better when one-step contextual problems were supported with informational graphics (Daroczy et al., 2015). Decorative and inaccurate graphic representations (Berends & van Lieshout, 2009; Boonen et al., 2014; Ott, 2020) and imprudent combinations of different representations (Elia, 2020) impair the solution process. Representation is an adequate model for a semantic structure of a problem if it can directly or interpretatively present all involved quantities whether known or unknown, the role of each quantity and their relationship and if it supports the solution process (Elia, 2020; Polotskaia & Savard, 2021; Savard & Polotskaia, 2017). Ott (2020) distinguished the drawability of relevant objects and relations in organisational graphics. For example, an arrow diagram and a combination of graphics and symbols are appropriate for representing the dynamic nature of *Change* problems, but less suitable for *Combine* and *Compare* problems (Elia, 2020; Verschaffel & De Corte, 1993).

2.1.3. Textbook Studies on Word Problems

Various studies confirmed that textbooks are prevalent instructional resources (Glasnović Gracin & Jukić Matić, 2016; Verschaffel et al., 2020). Assuming teachers choose tasks among those provided in textbooks and the choice affects students' opportunities to exercise in modelling, the question about the type of word problems in textbooks is legitimate. Previous studies showed that direct additive and subtractive *Change* problems and direct additive *Combine* problems are prevalent in textbooks, and problems with an operand unknown are rarely included (Singh et al., 2020; Verschaffel & De Corte, 1993). Singh et al. (2020) found that students were less successful in solving word problems that were underrepresented in textbooks. Vicente et al. (2022) compared types of word problems in Singaporean and Spanish textbooks regarding semantic structures and representations of word problems. They suggested that the function of graphics, *Organisational* versus *Decorative*, might better explain performance on international assessments than the quantity and variety of word problems.

2.2. Research Questions

Knowledge of textbook content is fundamental information on educational opportunities, and in particular when the analysis includes all offered textbooks in a country (Fan et al., 2013; Norberg, 2021). Previous research suggested that encountering a variety of word problems and representations contributes to students understanding and problem-solving abilities. It is important to examine the distribution of different types of word problems in different textbooks across all grades. This study aims to contribute to the classification of word problems by creating a methodological tool for task analysis. We state the following research questions:

1. Following the literature review, how can the classification of word problems be implemented for textbook analysis?
2. How are word problems, according to the suggested classification, arranged in the Croatian textbooks for primary mathematics education, across grades and editions?

3. Methodology

3.1. Context of the Study

In Croatia, teachers select an edition of textbooks among the ones preapproved by the Ministry of Education in four years cycle. The selection for the current four-year period is represented in Table 1. We examined textbooks from six different publishers across four grades of primary school in Croatia, a total of 24 textbooks. The edition excluded from the analysis is intended for the classes in special education. All textbooks need to be aligned with the national curriculum, hence they follow the prescribed learning outcomes and cover the same mathematical content. Textbooks differ in the organisation of units – ordering and naming, content within units – descriptions, characters, figures, tasks, additional materials – number of volumes, workbooks, worksheets, handbooks, and others. Following the definition of a word problem as related to the calculations with numeric values, we restricted the analysis in this study to the selected textbook units on arithmetic operations.

Table 1. The number of classes using each textbook in the period 2022. – 2026.

Textbook editions for each grade	1st grade	2nd grade	3rd grade	4th grade
Moj sretni broj	1378	1373	1338	1223
Super matematika za prave tragače	536	548	712	630
Otkrivamo matematiku	565	549	537	659
Matematika	206	249	240	240
Nina i Tino	264	198	238	262
Matematička mreža	184	172	187	175
Moja matematika	3	4	4	4

3.2. Textbook Analysis

This study is qualitative content analysis. It is a structuring procedure meaning the systematic coding evaluates the material against the deductively formed categories derived from theoretical consideration (Kuckartz, 2019; Mayring, 2015). In our study, a unit of analysis is a mathematical task in a textbook and it is assigned to categories based on the classification of word problems from the literature review. The textbook analysis consisted of several steps.

Step 1. Selection of the data and units of analysis

Table 2. Selected textbook topics for data analysis across grades.

1st grade	2nd grade	3rd grade	4th grade
1. Addition up to 5 2. Subtraction up to 5 3. Word problems 4. Relationship between addition and subtraction 5. Addition over 10 6. Subtraction over 10	1. Addition and subtraction of tens 2. Addition over tens 3. Subtraction over tens 4. Multiplication by 2 5. Division by 2 6. Multiplication and division by 9 7. Numeric expressions	1. Mental addition 2. Mental subtraction 3. Written addition over tens 4. Written subtraction over tens 5. Multiplication and division by decimal units 6. Written multiplication by 1-digit number 7. Written division by 1-digit number	1. Written addition 2. Written subtraction 3. Written multiplication 4. Written division 5. Numeric expressions

Data selected for this study included several different textbook units in each grade. Considering the roles of word problems as motivation and application for arithmetic operations, we mainly chose the first and the last unit related to each arithmetic operation in each grade, ensuring that each edition contains corresponding textbook units (Table 2). This choice is in line with the research questions because the presupposed criteria will be evaluated against different calculation contexts, mathematical situations, grades and editions. A unit of analysis is a separate mathematical task in the textbook unit. The data prepared for the analysis was a data matrix with columns for information and criteria and rows for tasks. Collected information included the capture of the task, textbook edition, grade, textbook unit title, page number and task number.

Step 2. Definition of categories

Categories for this study derived from the classification of word problems presented in the literature review. Word problems can be classified according to different criteria: their type, arithmetic and semantic structure, and given representation of the solution (Table 3). The categories in the criteria for the arithmetic and semantic structure are depended on other criteria. Contextual problems that have a one-step arithmetic structure can be further differentiated by their semantic structure. Contextual problems that have multi-step or equation structures cannot be categorised for their semantic structure in the same manner. Mathematical problems can have one-step or multi-step arithmetic structures, and the semantic structure does not apply to such tasks. For the representation criteria, the symbolic and graphic content accompanying written content in the given task and its solution were interpreted according to its function as given in the literature review.

Table 3. Categories in different criteria derived from the classification of word problems in literature.

Word problem type	**Arithmetic structure**	**Semantic Structure**	**Representation**
Not a word problem* Contextual Mathematical Authentic	Not a word problem* One-step – Result unknown – Operand unknown Multi-step – Result unknown – Operand unknown Equation Heuristics	Not a word problem* Additive/Subtractive – Change – Combine – Compare Multiplicative/Division – Groups – Rate – Compare – Array – Cartesian Other	Not a word problem* None Symbolic Graphic Decorative Informational Organisational Combination

*Category added in the trial run-through

Step 3. Trial run-through and coding rules

A trial run-through ensures the reliability and validity of the analysis. For the trial run-through, we chose the 2^{nd}-grade textbooks that include different calculations, hence there are various opportunities for the identification of structures and representations of word problems. All authors gathered and discussed coding each unit of analysis in the selected data upon reaching an agreement on the corresponding category. Where necessary, the categories were updated to be objective and comprehensive.

To complete the analysis, we added a category *Not a word problem* for each criterion, accounting for tasks not given in the written form, e.g. direct calculation with numbers or with a graphic model, and tasks given in written form but not related to calculations with numeric values, e.g. *Is it possible...?* For the arithmetic structure, the category *Result unknown* was differentiated into $a + b = \square$, $a - b = \square$, $a \cdot b = \square$, $a : b = \square$, and the category *Operand unknown* was differentiated into $\square + b = c$, $a + \square = c$, $a - \square = c$, $\square - b = c$, $a \cdot \square = c$, $\square \cdot b = c$, $\square : b = c$, $a : \square = c$. This is in line with the literature stating that the number of structures of word problems multiplies when the unknown changes. The category *Multi-step – Result unknown* was expanded with three sub-categories depending on the number of calculations required in the solution of the problem: 2, 3 or 4+ operations. For the mathematical problems in the semantic structure criterion, we added the categories for direct and multi-step calculation, and differentiating the latter according to the degrees of arithmetic operations required in the solution of the problem: 1st-degree, 2nd-degree or both degrees operations. We expanded the *Combination* category in the representation criteria to differentiate the three functions of the graphic content in the combined representation.

Step 4. Coding procedure

Table 4. An example of the data matrix and coding with assigned categories.

Example	**Capture**	**Edition**	**Grade**	**Unit title**	**Page, task**	**WP type**	**Arithmetic structure**	**Semantic structure**	**Representation**
A	 Onto the sum of numbers 41 and 25 add the number that has 3 tens and 2 ones.	M	2.	Zbrajanje dvoznamenkastih brojeva (33 + 25) [Addition over tens]	71, 13	MATH	MS-RU-2	MS-1ST	NONE
B	 Compare the differences and write the corresponding sign $<$, $>$ or $=$.	M	2.	Oduzimanje dvoznamenkastih brojeva (43 − 26) [Subtraction over tens]	83, 13	NW	NW	NW	NW
C	 Observe the figure. How many skaters are on the ice? Calculate by adding and multiplying.	M	2.	Množenje broja 2 i dijeljenje brojem 2 [Multiplication by 2, Division by 2]	26, 1	CONT	1S-RU $a \cdot b = \square$	MUL-GROUPS	COM-INF

Upon selecting a task, it needed to be identified as a word problem – a task set up in real-life, imaginary or mathematical context and solved using arithmetic operations, equations or heuristics on numerical data. Otherwise, the task was coded in the category *Not a word problem* (NW, see Example B in Table 4) for each classification criterion. A task that is a word problem was first assigned a word type category, then a corresponding category in arithmetic and semantics structure, and finally a category in the representation criteria. Example A in Table 4 is a mathematical word problem type (MATH), it has a multi-step two-operation arithmetic structure (MS-RU-2), all operations in the multi-step task are of the first degree (MS-1ST) and there is no accompanying representation (NONE). Example C in Table 4 is a contextual word problem type (CONT), its arithmetic structure is one-step multiplication with a result unknown (1S-RU $a \cdot b = \square$), and its semantic structure is multiplicative *Groups* problem (MUL-GROUPS). In this task, representation includes graphic content providing relevant information about the context and symbolic solution of the task, hence it is assigned to the *Combination* category, *Informational* subcategory (COM-INF).

Step 5. Category-based analyses

Four authors divided the units among themselves and proceeded with the content analysis separately. Each task in the textbook unit got assignation of category for every criterion – Word problem type, Arithmetic structure, Semantic structure and Representation. Any additional notes or issues about the context, structure, and representation were noted in the matrix and discussed among authors upon agreement. We used descriptive statistics and topic trace mapping (Schmidt, 1992) to analyse the data obtained from content analysis. First refers to determining frequencies and relative frequencies of content based categories across grades and editions. For the second method, the occurrence of a particular category was identified across units within a textbook. A devised diagram had textbook units ordered horizontally and word-problem categories labelled vertically. Each occurrence of a category, regardless of its frequency, had a mark in the diagram corresponding to vertical and horizontal diagram labels.

4. Results

4.1. Classification of Word Problems in Mathematics Textbooks for Primary Education

The proposed methodology for the task analysis yielded the expected results (Table 5). The number of word problems is nearly two-thirds of the number of non-verbal problems, and the number of contextual problems is almost five times the number of mathematical problems. The share of contextual problems is highest in 1st grade whereas the share of mathematical problems increases after the 1st grade. We found no examples of authentic problems in the analysed textbook units. Some of the tasks were problematic in the sense of an unrealistic context, e.g.

Example 8. On the first day 29 ants cleaned the anthill. On the second day, 42 more ants cleaned the anthill than on the first day. How many ants cleaned the hill on the second day?

The number of one-step problems with the result unknown is nearly thirty times the number of problems with the operand unknown, including contextual and mathematical problems. Hence, there are few opportunities to separate verbal cues from the specific arithmetic operation. Four-step problems and multi-step problems with the operand unknown are rare compared to the two and three-step problems. We found contextual problems set up as two or three separate tasks that are one-step calculation problems, e.g.

Example 9. Jack got 10 € *from his grandparents. He bought a book for* 7 €. *How much money has he got left?*

From the leftover money, he bought a comic book for 2 €. *How much money has he got left now?*

In the analysis, we categorised each one-step problem separately, e.g. Example 9 is coded as two one-step result unknown subtractive *Change* problems when it could have been formulated as a one two-step problem.

Table 5. Number of tasks in each category across classification and grades.

Criteria	**Categories**	**1st grade**	**2nd grade**	**3rd grade**	**4th grade**	**Total**
Word problem type	Contextual	106	119	110	70	405
	Mathematical	4	33	27	23	87
	Not a word problem	*144*	*236*	*242*	*125*	*747*
Arithmetic structure	One-step result unknown	104	105	96	64	369
	One-step operand unknown	1	7	3	2	13
	Two-step result unknown	3	25	23	15	66
	Three-step result unknown	1	12	12	9	34
	4+-step result unknown	0	1	2	3	6
	Multi-step operand unknown	1	2	1	0	4
Semantic structure	1st-degree *Change*	52	27	26	14	119
	1st-degree *Combine*	44	25	23	8	100
	1st-degree *Compare*	9	8	11	5	33
	2nd-degree *Groups*	–	33	9	7	49
	2nd-degree *Rate*	–	7	17	21	45
	2nd-degree *Compare*	–	3	5	0	8
	Direct calculation	3	9	8	11	31
	1st-degree operations	2	15	17	9	43
	2nd-degree operations	–	3	6	4	13
	Both degrees operations	–	22	15	14	51
Representation	Symbolic	0	3	0	0	3
	Graphic-Decorative	7	9	20	16	52
	Graphic-Informational	48	23	17	12	100
	Graphic-Organisational	5	10	5	1	21
	Combination-Informational	6	6	3	1	16
	Combination-Organisational	4	18	10	3	35
	None	40	83	82	60	265
	Total	254	388	379	218	1239

There were significantly more *Change* and *Combine*, and *Groups* and *Rate* problems, than *Compare* problems, especially multiplicative and division *Compare* problems. In the analysed textbook units, we found no multiplicative *Array* or *Cartesian problems*. This does not indicate that these types of problems are not included in the textbooks but that they are not common types of problems. Among multistep problems, 1st-degree and both degrees calculations dominate. To better understand this phenomenon, we additionally analysed multi-step contextual problems in 2nd grade by examining their complex semantic structure. Though it is possible to identify the underlining semantic structure of multi-step problems these tasks are not appropriate for categorisation as one-step tasks because there are numerous possible categories depending on the combination and order of semantics relations in the problem. The prevalent semantic clue in contextual multi-step problems in the analysed textbook units for 2nd grade is additive *Combine* structure, as in the question "How many altogether?". This verbal cue is used for itself, with 1st and 2nd-degree *Compare* structure and 2nd-degree *Groups* structure.

Table 6. The semantic structure of contextual multi-step problems in 2nd grade.

	Addition and subtraction	**Multiplication and division**	**Numeric expressions**
1st-degree ***Combine***	3		
1st-degree *Compare* and ***Combine***	2		
2nd-degree *Groups* and 1st-degree Change			1
2nd-degree *Groups* and 1st-degree ***Combine***		1	2
2nd-degree *Groups* and *Compare* and 1st-degree ***Combine***		1	
2nd-degree *Compare* and 1st-degree ***Combine***		1	1
2nd-degree *Rate* and 1st-degree ***Combine***		1	1
2nd-degree *Array* and 1st-degree Change			1
2nd-degree *Array* and 1st-degree ***Combine***			1

Graphic representations are almost exclusively related to contextual word problems. The function of the graphics changes across grades. In the first grade, it is mainly graphic informative as a replacement for extensive text. This function decreases but remains dominant in higher grades. The combination of symbolic and organisational functions of graphics is relevant in second grade; e.g. a picture of equal groups of items combined with repeated addition (see Example C in Table 4). Organisational graphics are mainly realistic visuals rather than diagrammatical. This explains fewer graphics with this function when working with larger numbers in higher grades.

4.2. Classification of Word Problems Across Grades and Editions

We calculated the relative frequencies of different types of problems across grades and editions (Table 7). For the first grade, contextual one-step problems make up

Table 7. Classification of tasks in two of the analysed textbook across grades.

Textbook A

Word problem type	Arithmetic structure	Semantic structure	1st grade	2nd grade	3rd grade	4th grade
Contextual	1 step-RU	1st-*Change*	12,50 %	7,41 %	16,67 %	9,09 %
		1st-*Combine*	12,50 %	7,41 %		
		1st-*Compare*	2,50 %	3,70 %	1,67 %	
		2nd-*Groups*		5,56 %	3,33 %	2,27 %
		2nd-*Rate*		1,85 %	5,00 %	13,64 %
		2nd-*Compare*		1,85 %	1,67 %	
	Multi-step	MS-1st		1,85 %	3,33 %	2,27 %
		MS-both				2,27 %
Mathematical	1 step-RU	Direct	5,00 %	1,85 %	3,33 %	13,64 %
	Multi-step	MS-1st		1,85 %	6,67 %	4,55 %
		MS-2nd		3,70 %	1,67 %	2,27 %
		MS-both		7,41 %	1,67 %	
Not a word problem			67,50 %	55,56 %	55,00 %	50,00 %

Textbook B

Word problem type	Arithmetic structure	Semantic structure	1st grade	2nd grade	3rd grade	4th grade
Contextual	1 step-OU	1st-*Change*				3,45 %
	1 step-RU	1st-*Change*	12,07 %	3,17 %		3,45 %
		1st-*Combine*	20,69 %	11,11 %	5,71 %	10,34 %
		1st-*Compare*	3,45 %	1,59 %	8,57 %	3,45 %
		2nd-*Groups*		6,35 %		
		2nd-*Rate*				3,45 %
		2nd-*Compare*			2,86 %	
	Multi-step	MS-1st	1,72 %			
		MS-both		1,59 %	7,14 %	3,45 %
Mathematical	1 step-OU	Direct		1,59 %		
	1 step-RU	Direct			1,43 %	10,34 %
	Multi-step	MS-1st	1,72 %	3,17 %		3,45 %
		MS-2nd			1,43 %	
		MS-both		4,76 %	4,29 %	3,45 %
Not a word problem			60,34 %	66,67 %	68,57 %	55,17 %

about one-third of all tasks for the majority of editions. The share of mathematical problems and contextual multi-step problems increases in higher grades for all editions. This is influenced by multi-step problems written as separate tasks in the first grade and the introduction of second-degree operations in the second grade. Three textbooks have around twice as many mathematical than contextual multi-step problems and two textbooks have four times as many contextual than mathematical multi-step problems. One-step problems with operand unknown are not common, and they appear more often as mathematical than contextual problems. Additive *Compare* and *Combine* problems and multiplicative *Groups* problems appear in almost all textbooks across all grades, thus these are the typical type of problems. Multiplicative *Rate* problems are more common in 3rd and 4th grade which can be explained with measurements introduced in these grades. Most textbooks have diverse contextual problems within grades.

Figure 1. Topic trace mapping diagram for two analysed textbooks.

Textbook C

	1st grade						**2nd grade**							**3rd grade**							**4th grade**				
CON-1st-*Change*	●	●	●	●	●		●	●						●	●						●	●			
CON-1st-*Combine*	●			●	●									●		●					●				
CON-1st-*Compare*				●	●	●			●								●			●		●			
CON-2nd-*Groups*											●													●	
CON-2nd-*Rate*										●		●						●	●	●			●	●	
CON-2nd-*Compare*											●														
CON-MS-1st							●							●		●									
CON-MS-2nd													●												
MATH-RU										●															
MATH-MS-1st							●								●										
MATH-MS-both																				●					●
MATH-MS-OU					●						●														
	1.	2.	3.	4.	5.	6.	1.	2.	3.	4.	5.	6.	7.	1.	2.	3.	4.	5.	6.	7.	1.	2.	3.	4.	5.

Textbook D

	1st grade						**2nd grade**							**3rd grade**							**4th grade**				
CON-1st-*Change*				●		●	●							●	●		●				●				
CON-1st-*Combine*				●	●			●								●	●				●				
CON-1st-*Compare*							●	●	●					●	●						●	●			
CON-2nd-*Groups*										●	●	●							●						
CON-2nd-*Rate*																		●					●	●	
CON-MS-1st									●						●	●									
CON-MS-2nd												●	●						●	●					
CON-MS-both																				●					●
MATH-RU							●			●				●											
MATH-OU											●			●											
MATH-MS-1st								●	●					●											
MATH-MS-both												●				●									●
	1.	2.	3.	4.	5.	6.	1.	2.	3.	4.	5.	6.	7.	1.	2.	3.	4.	5.	6.	7.	1.	2.	3.	4.	5.

When observing for the types of tasks within textbook units (Figure 1), we observed several potential patterns of task distribution. Majority of units include both contextual and mathematical word problems, except in the first grade where mathematical problems are not common. Arithmetic structures of the included tasks match the assigned arithmetic operation in almost every textbook unit. Additive and subtractive contextual, multi-step 1-st degree contextual and mathematical problems appear in units related to addition and subtraction, and multiplicative and division contextual, multi-step both degrees contextual and mathematical problems appear in units related to multiplication and division. The topic trace mapping diagrams for each textbook suggest that some patterns of task distribution could be characteristic for each edition. Textbooks might vary the types of word problems across units (see Textbook C for 2nd grade in Figure 1) or they might consistently use the same type of word problems. For the latter, in Textbook D for 2nd grade the observed pattern is: 1st-degree *Compare* and mathematical problems, then 2nd-degree *Groups* and mathematical problems. Some editions tend to include multi-step problems in every unit, and others mainly include them in units corresponding to the topic of Numeric expressions. Many units start with a contextual one-step problem accompanied with some graphics, organisational in lower grades and informational or decorative in higher grades (112/492). The problems and their solution are introductory examples for the textbook unit topic. Problems with operand unknown and multi-step problems mainly appear in later tasks within unit (110/492).

5. Discussion

The classification proposed in the methodology of the paper was exhaustive, unambiguous and appropriate for the analysis of word problems in primary mathematics textbooks regardless of grades and editions. Word problems were identified as tasks set up in real-life, imaginary or mathematical contexts, and solved using arithmetic operations on numerical data. Tasks were differentiated by four criteria: the type of context, arithmetic and semantic structure, and the function of graphics accompanying the task. Categories across criteria used for the content analysis are given in Table 6, and categories labelled with asterisks were not found in the analysed textbook units. The proposed classification was based on the literature review. Unlike other studies, we used multiple criteria in the analysis and considered multi-step problems.

In this preliminary study, we restricted our analysis to several textbook units. This is a major limitation for drawing general conclusions. However, the chosen units cover all arithmetic operations, hence the results are informative about implemented methodology and indicative of the distribution of different types of tasks.

A task can be classified into exactly one category for each criterion. The categories assigned for each task were appropriate for further analysis. Categories are complementary, i.e. categories for different criteria can be combined for a more elaborate classification. The results of the study suggest that textbooks include contextual problems with different semantic structures but that certain structures, 1st-degree *Change* and *Compare* and 2nd-degree *Groups*, are more common than others, that contextual problems with operand unknown are not common, that multi-step problems are more common as mathematical than contextual problems, and that underlying semantic structure of multi-step contextual problems is an additive *Combine* structure as in Example 4. The first finding is in line with the literature review, emphasising 1st and 2nd-degree *Compare* problems as particularly challenging. Fewer occurrences of such tasks and inadequate graphic representation might explain the challenge. The second and third findings indicate a lack of opportunities for encountering diverse modelling situations. Based on these results, we propose the following combinations of criteria:

- — word problem type versus arithmetic structure to check if one-step problems with operand unknown and multi-step problems are more common as mathematical than contextual problems in general;
- — arithmetic and semantic structure of contextual problems for each grade and edition to check the distribution of different types of tasks and identify learning opportunities for modelling different situations.

Table 8. Categories in different criteria proposed for the classification of word problems.

Word problem type	**Arithmetic structure**	**Semantic Structure**	**Representation**
Not a word problem Contextual Mathematical Authentic*	Not a word problem One-step – Result unknown ($a+b=\square$, $a-b=\square$, $a \cdot b=\square$, $a:b=\square$) – Operand unknown ($\square+b=c$, $a+\square=c$, $a-\square=c$, $\square-b=c$, $a \cdot \square=c$, $\square \cdot b=c$, $\square:b=c$, $a:\square=c$) Multi-step – Res. unknown 2-steps – Res. unknown 3-steps – Res. unknown 4+ steps – Operand unknown Equation* Heuristics*	Not a word problem Additive/Subtractive – Change – Combine – Compare Multiplicative/Division – Groups – Rate – Compare – Array – Cartesian* Direct calculation Multi-step – 1st-degree operations – 2nd-degree operations – Both degrees operations Other*	Not a word problem None Symbolic Graphic – Decorative – Informational – Organisational Combination – Decorative – Informational – Organisational

Categories of semantic structures can be implemented for the analysis of multi-step contextual problems with an extensive report.

The results of the study indicate that the number of graphic representations decreases over grades and its function changes from *Informative* to *Organisational* to *Decorative*, which is in line with curriculum requirements and developmental age. The graphics in the analysed textbook units with *Organisational* function use realistic visuals of the context rather than diagrammatical. This collides with the findings of other studies arguing for adequate diagrammatical representation rather than decorative representation. Further detailed analysis of graphic representations should additionally differentiate between diagrammatical and realistic visual graphic representation.

Textbooks seem to follow certain patterns of task distribution. The motivation and concretisation role of one-step contextual problems, as introductory examples, is emphasised, and henceforward the "new" calculations are used in more complex word problems, problems with operand unknown, mathematical problems, and multi-step problems, emphasising their application role. The pattern of consistent use of the same types of tasks, the emphasis on certain semantic structures of one-step contextual problems, and fewer share of operand unknown and multi-step contextual problems reduce the modelling role of word problems.

The issue of theme in contextual problems arose in our study, but its exploration is beyond the scope of this paper. Further study could focus on the reality of context, the relevance of themes regarding socio-cultural-economic questions, students' interests etc., and the connection of themes across units and edition or other questions.

6. Conclusion and Suggestions

Development of students' skills in solving word problems is influenced by the type of given problems, their arithmetic and semantic structure, and interpretation and treatment of problems (Sianturi et al., 2021; Verschaffel et al., 2020; Vicente et al., 2022). Providing diverse opportunities in working with contextual problems is relevant knowledge for teachers, textbook authors, scientists and policymakers. Teachers can advert to different textbook editions and resources to choose or design a variety of word problems. Textbook authors can check the diversity of offered word problems against the proposed framework. Policymakers can emphasise and encourage working on authentic, heuristic word problems. This study raised several issues and further research following the proposed methodology should be undertaken.

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Tekstualni zadaci u udžbenicima za razrednu nastavu matematike u Republici Hrvatskoj

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Sažetak. Tekstualni zadaci su neizostavan dio matematičkog obrazovanja. U literaturi su to zadaci postavljeni u stvarnom, imaginarnom ili matematičkom kontekstu koji se rješavaju pomoću aritmetičkih operacija na numeričkim podacima. Tekstualni zadaci se mogu klasificirati prema različitim kriterijima: vrsti konteksta, aritmetičkoj i semantičkoj strukturi te funkciji grafike koja prati zadatak. Proveli smo analizu sadržaja odabranih udžbeničkih jedinica pokrivajući sve računske operacije u šest udžbeničkih izdanja i sva četiri razreda osnovnog obrazovanja. Rezultati su zato informativni o primijenjenoj metodologiji i indikativni o raspodjeli različitih tipova zadataka. Klasifikacija predložena u ovom radu iscrpna je, nedvosmislena i prikladna za analizu tekstualnih zadataka u udžbenicima matematike za niže razrede osnovne škole bez obzira na izdanje i razred. Rezultati studije sugeriraju da su određeni tipovi kontekstualnih zadataka češći od drugih i da su problemi koji imaju aritmetičku strukturu s nepoznatim operandom ili problemi s više koraka češće matematički nego kontekstualni. Pružanje različitih prilika u radu s kontekstualnim problemima relevantno je znanje za nastavnike, autore udžbenika, znanstvenike i kreatore politika, stoga bi trebalo poduzeti daljnja istraživanja slijedeći predloženu metodologiju.

Ključne riječi: tekstualni zadatak, kontekstualni problem, grafički prikaz, osnovnoškolsko matematičko obrazovanje, matematički udžbenik, analiza sadržaja

Teaching Mathematics to Preschool Children through Elements of Performing Arts

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Abstract. The aim of this paper is to connect mathematics and the elements of performing arts (dance, movement, play, music, and the like) in order to make certain mathematical content more familiar to the children of early and preschool age, and make it more entertaining, thus making learning of mathematical content easier for them in general. Current practice clearly shows that children of early and preschool age are not familiar enough with basic mathematical content. One of possible reasons is that educators are, generally speaking, methodologically not prepared sufficiently for early mathematics, which implies the necessity of changes in curriculum, namely, in very approach to and way of working with children. Therefore, the emphasis is placed on chronologically appropriate teaching, as well as on the importance of initial understanding and adoption of mathematical logic and concepts. Concrete mathematical examples are elaborated in this paper with the aid of performing elements in order to enhance the current curriculum. Mathematical and performing procedures will be applied to the examples of learning and comparing numbers, sets, and relations of sets and basic geometric content.

Keywords: mathematics, performing arts, preschool age, learning

1. Introduction

Institutions of early and preschool education are focused on holistic development of children (emotional, social, cognitive, development of thinking, imagination, speech, etc.). Early childhood is a sensitive period in which intensive changes occur, and the child is under influence of his environment. The whole early and preschool period is marked exactly by children's curiosity, experimenting with various materials, fun, freedom, imagination and creativity. These are the reasons why early and preschool period is optimal for presenting mathematical concepts through dance and movement. Relating art and science has proved as an excellent symbiosis.

In their book "Drama-Based Pedagogy", Dawson and Lee emphasize that teaching is a process that should be reflexive, active and cyclic, and by no means linear. They are answering questions: "What makes an environment effective for learning? What can teachers (educators) do to enhance pupil's/child's engagement, to increase cooperation, participation, and to encourage development of critical thinking and creativity within each pupil or child?" (Dawson & Lee, 2018, p. 5). They find the answers to these questions exactly in symbiosis of art and science, relying on numerous studies which highlight the advantages of art, especially drama as interdisciplinary approach to social, economic and academic learning. By merging education, drama theory and praxis, we can discover potential of art as a pedagogic tool (Dawson & Lee, 2018). What is essential are well-thought ideas realized as performing acts with simple props supported by educators' quality animation in order to achieve excellent results. It is highly important to get children interested, to encourage them to play and be creative in order to adopt and develop critical thinking towards a learning object.

Gelman thinks that the urge to learn is innate category and he imagines small children "as self-controlling learning machines who tend to learn on the fly, even when they are not at school and without help of adults" (2000, p. 31). The famous Swiss psychologist Piaget classified human intellectual development into four periods. According to this theory of intellectual development, children of early and preschool age (18 months to 7 years) belong to the intuitive period. The first part of this period is up to the fourth year, and during this period the child's perception is well developed. "This period marks the growth of the child's ability of describing, and is clearly manifested in the choice of words when describing objects (a ball, for example), actions ("go", "do") and relations between objects" (Liebeck, 1995, p. 70). In the second part of intuitive period, according to Piaget, the child's view of the world is egocentric. In the end of this period, the child begins to think logically. He believes that intellectual development is equal for all children whereas a great number of psychologists do not agree with this. Bruner is one of them. According to him, the intellectual development of children can be influenced by learning (Bruner, 1968). Thus, it is important to introduce simple mathematical concepts at an early child's age.

Balfanz (1999) deems that mathematical education is a key part of kindergarten praxis, but it is also a key for later success in mathematics (Bowman et al., 2001; Campbell et al., 2001; Reynolds & Ou, 2003). Numerous scientists proved that mathematical education of small children and preschool age children increased their self-confidence, and facilitated mastering mathematical content later in life (Stipek & Ryan, 1997). Kindergarten teachers play an important role in the development of mathematical skills. In order to achieve this, educators must be aware of the intellectual development of children. They are expected to develop their own creativity, knowledge and education in the first place, in order to be able to recognize and develop all of these in children. Educators in kindergartens should encourage children to develop their mathematical skills through various games and workshops which contain different objects and materials enabling children to work with them (Ginsburg et al., 2006).

Children can learn mathematics through play, they create mathematical ideas, explore shapes and symmetry (Seo & Ginsburg, 2004). "Play is an unquestionable activity for the period of early and preschool education. It is a method, strategy and educational philosophy of a child of that age" (Šandor, 2015, p. 6). During free play, educators pay little or no attention to mathematics (Lee, 2004). It seems that educators are insufficiently trained to see the possibility of teaching a range of mathematical concepts through play (Moseley, 2005). Many studies show that kindergarten teachers do not like mathematics, so they do not like to teach it (Kowalski et al., 2001; Lee, 2004). Most educators teach preschool children mathematics through very small content range (Graham et al., 1997). They fail to encourage counting and estimating enough, and they rarely use mathematical terminology (Frede et al., 2007). Children are capable of adopting more complex content than mentioned. The awareness of this problem is precisely the impetus for writing this paper through joining mathematics with drama and performing elements to help educators in teaching early mathematics.

2. Mathematical concepts in early education

Mathematics is unavoidable for a child's cognitive development, and children acquire mathematics every day through their natural environment (Ginsburg et al., 2006). Children are capable of adopting numbers and some geometric content they usually meet from an early age (Geary, 1996). According to Liebeck (1995), children learn numbers, operations and relations with numbers through sorting, joining, matching, sequencing. The adoption of numbers, numerical operations, and relations with the numbers is conditioned by the adoption of the concept of a set (Ginsburg et al., 2006). From early childhood, children develop different ways of counting (Groen & Resnick, 1977), distinguish sets with different number of elements (Lipton & Spelke, 2003) and understand basic ideas of addition and subtraction of numbers up to five by means of the fingers of one hand (Brush, 1978). Preschool children do not understand why $3+4=7$, but by adding four candies to three candies, they know that they get seven candies. They know that by adding we get more, and by subtracting we get less (Ginsburg et al., 2006). According to Liebeck (1995), geometric shapes that children might adopt in kindergarten are circle, square, rectangle and triangle. Children discover objects in the space in which they live (Newcombe & Huttenlocher, 2000) and relations in space, but they can't recognize obtuse triangle, and for them the triangle is exclusively pointed (Clements, 1999).

In this paper, we present mathematical exercises designed for children of early and preschool age aimed at adopting basic mathematical concepts in visual and effective way by trial and error.

3. Research Methodology

Researc|h subject matter: In our research, we aimed to check whether children of preschool age are capable of adopting and relating more complex mathematical concepts such as numerical relations and corresponding signs, set relations, line segment, line and half-line, and geometric shapes. These concepts have crucial importance for further adoption of mathematical skills. The concepts and shapes were represented using simple colored ribbons. We animated some children as creators, and children from the "audience" recognized and distinguished the displayed figures.

Researc|h sample: Exercises demonstrated in this paper were performed in the course of one month in several institutions of early and preschool education (Kindergarten Sportić in Mostar, Kindergarten Čitluk in Čitluk) in the area of Herzegovina-Neretva Canton. The research was conducted in 2023 and 52 children took part in it. Exercise tasks were specifically created and devised for children of higher preschool age (chronological age of children participating in the research was five). They were focused on adopting mathematics concepts through four areas, namely:

— memory (animating children of early and preschool age)

— remembering (revision of exercises through dialog and group's logical thinking)

— reproduction (repeat exercises through playing emphasizing the game winner)

— recognizing (relate mathematical concepts to individual completed exercise)

4. Demonstration of mathematical-performing exercises

In this section, we demonstrate three conducted mathematical exercises.

Exercise 1. Learning and comparing numbers

The first exercise related to counting and comparing groups whose members were the children. In order to motivate the children to cooperate, we explained the game rules and gave them instructions. The children actively and willingly participated in the implementation of the assigned exercises. Figure 1 shows mathematical comparing of numerically different sets of children who simulated members of sets and mathematical relational signs (equity or inequity) with their bodies, which is a feature of stage, i.e. performing act. After explaining and animating the children, they delightfully accepted the "game" and kept repeating it without the educators' help.

Figure 1. Children's performing movements.

Exercise 2. Sets and relations between sets

The second exercise consisted of two parts. The first part related to sets, relations between sets and determining members of sets. The children who watched the exercise were assigned to count their friends within each group, and to name who belonged to a purple group and who belonged to a yellow group (Figure 2). With this part of the exercise, we also achieved the children's recognition of groups with equal number of members, which was the goal of the first exercise.

Figure 2. Children in groups.

In the second part of the exercise, the children also formed two overlapping groups, purple and yellow (Figure 3). The task of this exercise, for the observing children, was to name and count their friends in purple and yellow groups, and then to compare the groups. After the animation and walking along the purple ribbon, we concluded together that Marko, Ivan and Fran were in the purple group (the names do not correspond to real people), and it counted three of their friends.

Figure 3. Counting and group belonging.

We did the same with the yellow ribbon and concluded that the yellow group included Ivan, Fran, Luka and Tomislav (the names do not correspond to real people), and the yellow group included four of their friends. In the end, we concluded that both Ivan and Fran belonged to both the purple and yellow groups. The aim of this exercise was to name the boys who were in two groups at the same time, as well as the new group bounded by the purple and yellow ribbons, which consisted of two boys.

After familiarizing children with this game, the “observing” children had no more problems counting members of both purple and yellow groups, comparing them and answering a very important question in the end, namely, who belongs to one group, who belongs to the other, and who makes the new group and how many groups are there now.

Exercise 3: Basic geometric content

The third exercise, which relates to geometric content, was performed in three parts. The first part of the exercise related to showing spatial geometric content by means of rope as a prop, and animating children to be simulators of points. We used the rope for teaching the basic concepts of the line, half-line and line segment. In order to understand the previously mentioned concepts, we stretched three ropes (Figure 4). The first rope was unattached to anything and it represented the line. We said for the second rope with a girl sitting on one of its ends that it represented the half-line, and we positioned two boys on the ends of the third rope to teach the concept of the line segment. For the purpose of children adopting the previous concepts, we clearly showed that the line has neither beginning nor the end, that the half-line had the beginning but not the end, and line segment had both beginning and the end. The initial teaching was followed by practice of pronouncing their names, and by performing movement of sitting (standing) we matched each concept with its layout.

Figure 4. Line segment, line and half-line.

The second part of the exercise was devoted to creation and recognition of geometric shapes. Using ribbons of different colors, the children formed rectangle, square, triangle and pronounced names of created geometric shapes (Figure 5). Besides forming a rectangle, square and triangle, the aim of this exercise was to recognize whether they are made of lines, half-lines or line segments.

Figure 5. Rectangle, square and triangle.

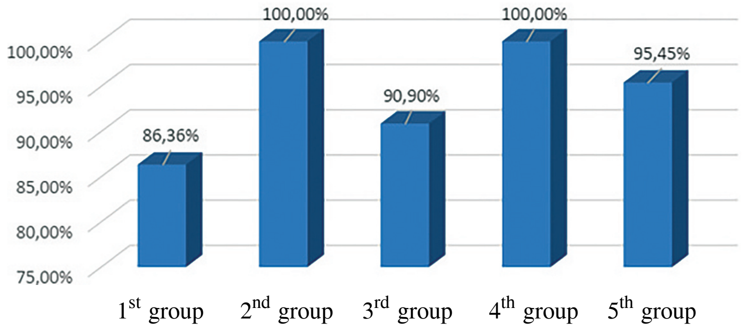
The third part of the exercise relates to the introduction to and distinguishing between the circle and circle disc. The exercise was presented by means of a hula hoop. Through play, the children learned the terms for circle and circle disc, becoming aware that a hula hoop itself makes a circle whereas a hula hoop and everything inside it makes a circle disc. The exercise is repeated through playing "jump into the circle", "jump out of the circle" (Figure 6).

Figure 6. Circle and circle disc.

5. Reflection

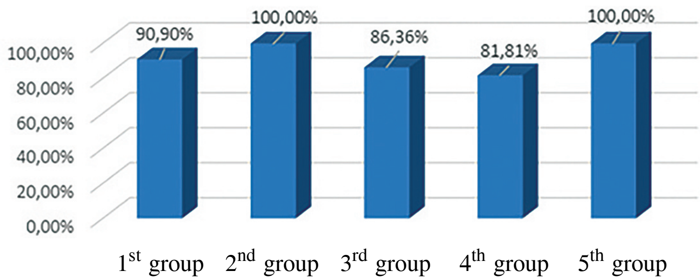
It is known that playing is a free act and that man in his existence is *homo ludens*, that is, a being of play, which was confirmed by authentic animation of children in kindergartens and resulted in spontaneous, natural content adopted through play. To adopt the mentioned exercises better, we divided the children in five groups. Three groups were made of ten children each, and two remaining groups consisted of eleven children each. Our animation of earlier performed demonstrated exercises was followed by daily repetitions of the exercise during one month (22 workdays), and recognition of individual mathematical terms as well as recording of the results achieved. The results are a product of a reply of a group representative in agreement with other group members. In the next part, we are providing the results of the adoption of performed exercises.

In the first exercise, the children were learning about numerical relations and their meanings. Each group results for this exercise are provided in Graph 1. It shows that the average group numerical relations adoption is 94.54 %, which indicates that the children of this age are excellent in spotting the relations of "fewer ($<$)", "more ($>$)" and "equal ($=$)" when counting and comparing clearly presented sets.



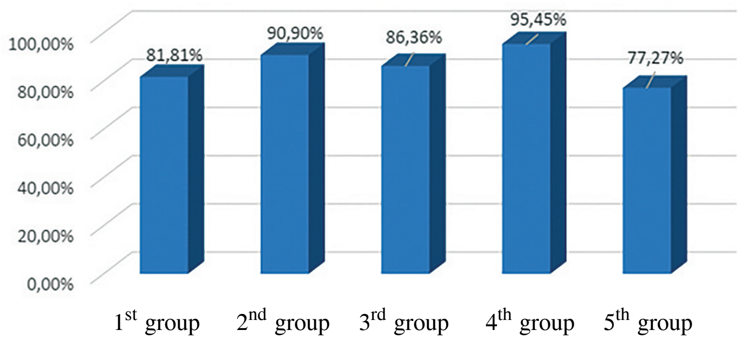
Graph 1. Numerical relations adoption percentage.

The results of the second exercise which related to counting and group belonging are shown in Graph 2. The conclusion made after several days of Exercise 2 repetition is that set recognition and counting adoption stacked up to 91.81 %, which suggests very good results in the adoption of this content.



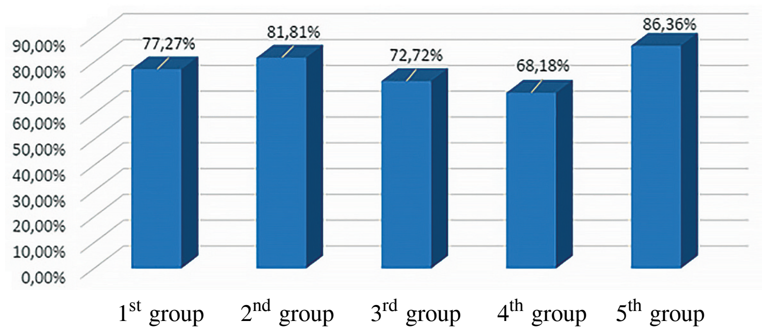
Graph 2. Counting and group belonging adoption percentage.

The outcome of the third exercise results is proper understanding, recognizing and distinguishing between the line, half-line and line segment (Graph 3). The average adoption of content relating to the line, half-line and line segment is 86.36 %. The reason of somewhat lower percentage compared to the previous exercises is in seeing the rope as a line segment by children in some groups. The conclusion is that children understand the concept of line harder, probably due to its endlessness.



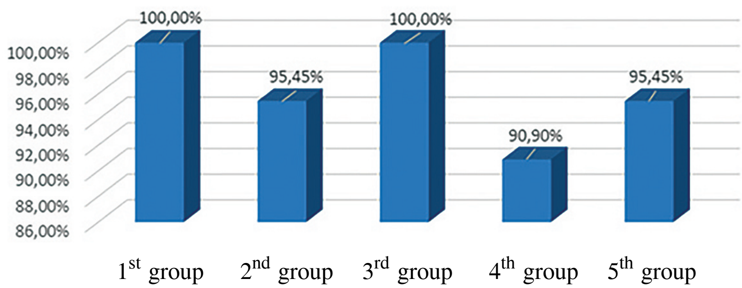
Graph 3. Line, half-line and line segment adoption percentage.

In the second part of this exercise the children were forming and recognizing geometric shapes: the square, rectangle and triangle. The results of this exercise are shown in Graph 4. Percentage of 77.27 % is a result which is in some way understandable as children of this age confuse square and rectangle. This could be expected as they daily meet these shapes but fail to identify them by those names, unlike the triangle.



Graph 4. Geometric shapes adoption percentage: triangle, rectangle and square.

The third part of this exercise, the circle and circle disc recognition, did not represent a problem to the children (Graph 5). Through play and competitive spirit, the exercise resulted in successful understanding of concepts of the circle and circle disc. These concepts were adopted by 96.36 % of the children.



Graph 5. Circle and circle disc adoption percentage.

6. Conclusion

Didactic tools for introduction of preschool age children into the world of mathematics are demonstrated in this paper. Concrete mathematical examples were elaborated by stage elements in order to enhance the existing curriculum. The children simulated set elements, both equality and inequality signs, lines and shapes using their bodies. Mathematical and performing procedures were applied to the following examples: learning and comparing of sets, relations between sets, and basic geometric content.

Simple props such as rope, colored ribbon and hula hoop were used. By animating their own bodies, the children took part in stage performing, namely displaying mathematical concepts and relations between them. They were separated into groups of either "performers" or "observers", which initiated their interaction and realization of certain exercises. The exercises were constructed in such a way as to build on one another, so that the previously mentioned mathematical concepts would find their place in the following exercise, and the children would recognize them easily. At the end of performed exercises, our recommendation to educators of the children who took part in these activities is to continue implementing these exercises so that their efficiency would lead to recognition of mentioned mathematical concepts.

The result of connecting mathematics and performing (stage) arts turned out to be extremely successful and it confirmed the potential of arts as pedagogic tool.

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Matematičko poučavanje djece predškolske dobi kroz elemente scenske umjetnosti

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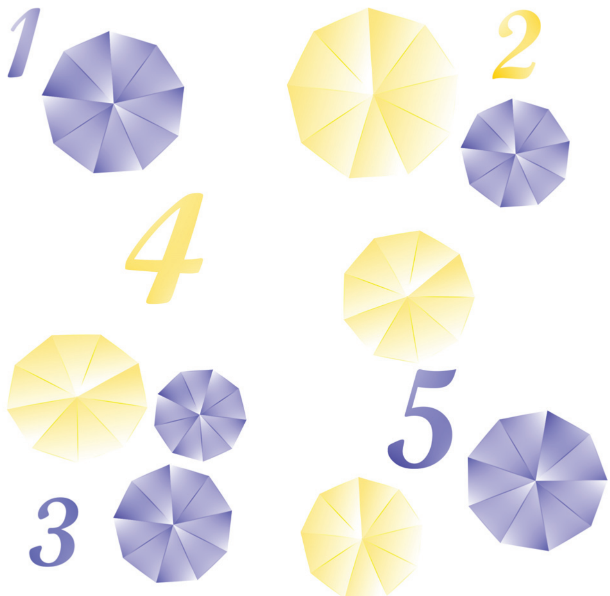
Sažetak. Cilj ovog rad je povezati matematiku i elemente scenske umjetnosti (ples, pokret, igra, glazba i sl.) kako bismo djeci rane i predškolske dobi približili određeni matematički sadržaj i učinili ga zabavnim, olakšavajući im na taj način općenito učenje matematičkih sadržaja. Iz dosadašnje prakse, razvidno je kako djeca rane i predškolske dobi nisu u dovoljnoj mjeri upoznata s osnovnim matematičkim sadržajima. Jedan od mogućih razloga je što odgajatelji općenito nisu metodološki dovoljno pripremljeni za ranu matematiku, što implicira nužnost promjena u kurikulumu odnosno u samom pristupu i načinu rada s djecom. Stoga naglasak je stavljen na kronološki prikladnom poučavanju kao i važnosti inicijalnog razumijevanja i usvajanja matematičke logike i pojmova. U radu su razrađeni konkretni matematički primjeri pomoću scenskih elementa kako bi se unaprijedio postojeći kurikulum. Matematičko–scenski postupci primjenjivat će se na primjerima učenja i uspoređivanja brojeva, skupova i odnosa skupova i osnovnog geometrijskog sadržaja.

Ključne riječi: matematika, scenska umjetnost, predškolska dob, učenje

3.
Exploring Geometric Concepts and Spatial Abilities in Mathematics



Collective Classroom Social Climate in Croatian Geometry Lessons: An Analysis of Children's Drawings

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Abstract. The classroom is a social context in which students act and learn on a daily basis. Over time each classroom develops a specific social climate with different characteristics. Since the classroom climate refers to specific curricular and pedagogical components, it may differ between different school subjects as well as between topics (e.g., within mathematics). The goal of this paper was to examine the collective classroom social climate as presented in students' drawings of their geometry lessons. The classroom social climate was analysed on the basis of three categories, namely Interpersonal Relationship, Personal Growth, and Order, in six elementary classes of the same school (two classes per Grades 3–5). The results showed some similarities and differences among the participating classes. The differences were reflected in teachers' enactment (the teacher makes mathematics statements or asks mathematical questions), students' enactment (students discuss or listen passively), and teaching materials and tools specific for geometry lessons (e.g., 2D-shapes and models, 3D-solids and models, geometric tools). The results showed that the majority of classes, regardless of the grade level, presented geometry lessons as being taught frontally with the teacher standing in front of the blackboard and students at their tables in the traditional seating arrangement. The goal of the lesson was clearly shown in almost all of the drawings. In a broader sense, the similarities in classroom climate seen across classes and grade levels may also be indicative of aspects of the school climate in geometry lessons.

Keywords: social climate, collective climate, geometry, elementary school, primary education

1. Introduction

Social climate is usually described as the perception of the social environment shared by a group of people (Bennett, 2010). Social climate in education is shaped by the relationships between teachers and students, and between students. Students' self-concept, motivation, and performance are further influenced by the quality, quantity, and direction of these relationships (Fraser, 1983). The concept of social climate is related to school climate, school ethos and classroom climate (Allodi, 2010). The term "school climate" refers to the set of internal characteristics of the school, and these characteristics distinguish one school from another, and affect the behaviors of each member of the school. Furthermore, school climate is a quality of the school environment which is experienced by participants, it has an impact on their behavior and is grounded on their collective perceptions of school behavior (Hoy & Miskel, 2005). The school climate includes subjectively significant characteristics that relate to the whole school as an organizational unit: the physical environment of the class, the social relationship between students and teachers, expectations and practices regarding social behavior, order and discipline, performance expectations (the way in which the teaching process takes place) and cultural self-image (Eder, 2002). According to Brookover at al. (1978), some aspects of the school social climate affect the academic achievement of schools. The school climate depends on specific situations, which are different for each school, namely "It is conditioned by the school environment, where we can observe it, analyze it, and evaluate it. It does not just happen but grows continually. It is a long-lasting phenomenon which is typical of the individual school" (Kantorová, 2009, p. 184).

As mentioned above, social climate can be also looked at from the perspective of a classroom given that the classroom is a social context in which students live and learn daily. Over time, each class develops a social climate with specific characteristics (Evans et al., 2009; Moos & Moos, 1978; Trickett & Moos, 1973). In order to better understand, as well as to improve, classroom climate, many studies were conducted (Anderson et al., 2004). These gained popularity due to the discovery of a relationship between academic motivation, participation and engagement, and positive classroom climate (Evans et al., 2009).

As a result of the research interest, considerable progress has been made over the last twenty years in terms of measuring, analyzing, and conceptualizing classroom climate. Over the years a number of different instruments have been used in order to examine students' perceptions of the classroom climate, but these mainly focused on the secondary level. Kuzle and Glasnović Gracin (2019, 2021) adapted the existing models and tests, such as the Classroom Environment Scale ([CES]) by Trickett and Moos (1973) (for more detail see Kuzle and Glasnović Gracin, 2021) to develop a model to research primary grade students' perceptions of classroom social climate using children's drawings. Due to its detailed structure, which includes domains, dimensions, and subdimensions with accompanying scales, it has proven to be a valid model for capturing classroom social climate reflecting versatile behaviors, actions, situations, and experiences (Kuzle, 2023).

The main goal of the inquiry presented in this paper was to gain insight into the collective classroom social climate in geometry lessons in six classes from one school (two classes each from grades three to five) using participant-produced drawings (Kearney & Hyle, 2004). To achieve this, the students' data were analyzed using a model and an analysis instrument of classroom social climate for primary grade students (Kuzle & Glasnović Gracin, 2021) which is an adaption of the CES (Trickett & Moos, 1973). Given the fine structure of the model including domains, dimensions, and subdimensions with accompanying scales, we provide characteristics of collective classroom social climates by looking at each class as a collective.

2. Theoretical background

In this section, we first present the concept of classroom social climate, followed by the classroom social climate model of Kuzle and Glasnović Gracin (2019, 2021) used in this study. We also introduce drawings as a research method. The section ends with the two research questions that guided the study.

Classroom social climate model

Each student participates in the classroom and has an impact on the classroom social climate (Evans et al., 2009; Moos & Moos, 1978; Trickett & Moos, 1973). Evans et al. (2009) describe classroom social climate as "an important construct for effective schooling" (p. 141). Each class develops an individual social climate over time which consists of various factors such as rules, style of leadership, and student task-related interaction (Moos & Moos, 1978; Trickett & Moos, 1973). Teachers often speak about classroom climate, or classroom environment, atmosphere, or ambiance, which they consider to be essential for good learning outcomes. In the last few years, many different classroom climate models have been used to examine perceptions of the classroom environment (Fraser, 1983). According to Evans et al. (2009), the reason studies on classroom climate became so popular is because of the relationship between educational outcomes and positive classroom climate. Furthermore, the classroom climate can have an impact on student development and growth (Moos & Moos, 1978).

Kuzle and Glasnović Gracin (2021), based on the existing CES instrument (Trickett & Moos, 1973), conceptualized classroom social climate through three conceptual categories: *Interpersonal Relationship, Personal Growth, and Order*. Each category consists of different dimensions and subdimensions with accompanying scales (Kuzle & Glasnović Gracin, 2019, 2021), as presented in Figure 1. According to Kuzle and Glasnović Gracin (2019, 2021), the first category *Interpersonal Relationship* refers to the type and intensity of personal relationships, as well as the teacher-student interaction in the classroom which includes social, pedagogical, and mathematical aspects. The *Interpersonal Relationship* category is divided into three dimensions, namely *Verbal and non-verbal communication of the teacher*, *Verbal and non-verbal communication of the students*, and *Organization*. *Verbal and non-verbal communication of the teacher* is divided into two subdimensions: the teacher's position in the classroom and teacher support. *Verbal and non-verbal communication of the students* is described by three subdimensions, namely, the students' position in the classroom, participation, and affiliation.

Organization, the third dimension of the *Interpersonal Relationship* category is divided into two subdimensions: working method, and classroom seating arrangement. The second category *Personal Growth* refers to specific opportunities for learning mathematics with respect to the goals and clarity of the lesson objective as well as teaching resources. This category is described by two dimensions: goal orientation and teaching materials and tools. The third category *Order* refers to the social norms and maintaining order in the classroom, it is described by one dimension, namely keeping order. Kuzle and Glasnović Gracin (2019, 2021), and Kuzle (2023) used this model and accompanying analytical tool in their studies on the characteristics of the classroom social climate in Grades 3 to 6. Here, all categories and dimensions of the model and the tool were presented in students' drawings. Thus both the model and the instrument have proven to be theoretically coherent and suitable to understand how young students view the psychosocial characteristics of a mathematics learning environment.

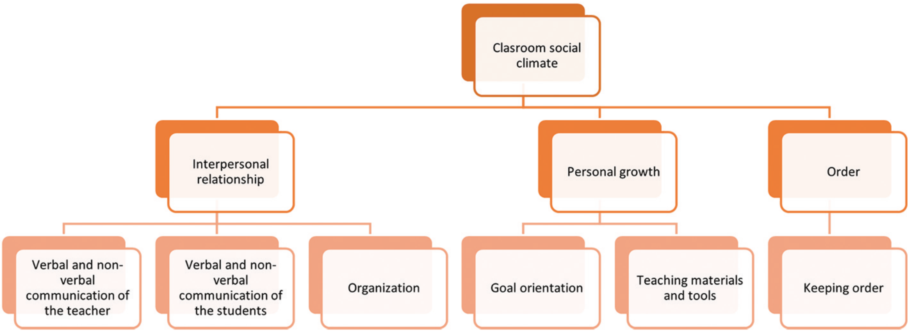


Figure 1. Conceptualization of classroom social climate (Kuzle & Glasnović Gracin, 2021, p. 748).

Drawings-based research

Drawing is an effective way to express one's thoughts, clarify them for others and symbolize them (Anning, 1997). Not only does a young child's drawing provides insights into his/her perception of the world and their relationships with important people, objects, and places around them, but it can also offer an insight into young children's process of thinking (Anning & Ring, 2004). Many researchers (Ahtee et al., 2016; Kuzle, 2023; Kuzle & Glasnović Gracin, 2019, 2021; Laine et al., 2015; Pehkonen et al., 2011) have used drawings as a method in research with young children. Researchers can gain insight into children's thoughts and perceptions through their drawings. Also, with young children, it is more appropriate to use drawings than, for example, questionnaires, interviews, and observations (Ahtee et al., 2016; Anning, 1997; Einarsdóttir, 2007). Compared to traditional data collection techniques, students' drawings demonstrate significant benefits in qualitative research, including familiarity with the act of drawing and non-verbal communication, as well as overcoming the potential language barrier and keeping language mediation to a minimum, which is especially useful when working with young children (Ahtee et al., 2016). Moreover, according to Einarsdóttir (2007), they can help students in recalling and expressing additional details about the events they have depicted. Thus, drawings are considered to be highly rich sources of information for understanding children's perceptions of teaching.

In the last few years, researchers (Ahtee et al., 2016; Kuzle, 2023; Kuzle & Glasnović Gracin, 2019, 2021; Laine et al., 2015; Pehkonen et al., 2011) have used drawings as a data collection instrument in order to study students' perceptions of mathematics (specifically geometry), to gain insight into how students perceive mathematics lessons, and to investigate their beliefs, attitudes and emotions concerning mathematics. In their work, Laine et al. (2015) examined what kind of emotional atmosphere dominates in Finnish third and fifth graders' mathematics lessons. In order to see the characteristics of the collectives, the participants were students from a number of different classes. Ahtee et al. (2016) used students' drawings to investigate teachers' and students' activities during a mathematics lesson. Pehkonen et al. (2011) used drawings to explore students' conceptions of mathematics. The focus of their study was to analyze teacher-student communication. Thus, many studies (e.g., Ahtee et al., 2016; Laine et al., 2015; Pehkonen et al., 2011) used drawings as a research method in order to examine the social and emotional characteristics of a mathematics lesson in general. In contrast, there are only a few studies that focus on a specific area of mathematics pertaining to aspects of classroom climate, such as geometry (Kuzle, 2023; Kuzle & Glasnović Gracin, 2018, 2019; Kuzle et al., 2018).

The aim of the research and research questions

The aim of this study is to examine the collective classroom social climate of geometry lessons in primary education in the Republic of Croatia by using a multidimensional model developed by Kuzle and Glasnović Gracin (2019, 2021). The following two research questions guided the study:

1. What features of collective classroom social climate in geometry lessons are predominant in different primary school classes?
2. How does the collective classroom social climate in geometry lessons differ between different primary school classes?

3. Method

3.1. Study design and sample

For this study, an explorative cross-sectional qualitative research design was chosen (Patton, 2002) using participant-produced drawings (i.e., students' drawings and students' interviews) (Kearney & Hyle, 2004). The study participants were 85 primary school students (Grades 3–5) from the same elementary school in Zagreb (Republic of Croatia). The school was chosen based on its previous cooperation with the third researcher's university. Six different classes and three different grades from the same school participated in this study. The distribution of students among the different grades and classes is shown in Table 1. While classes 5.a and 5.c were taught by the same mathematics teacher, 3.a, 3.b, 4.a and 4.c each had their own class teacher for all subjects, including mathematics.

Table 1. The distribution of students among the different grades and classes.

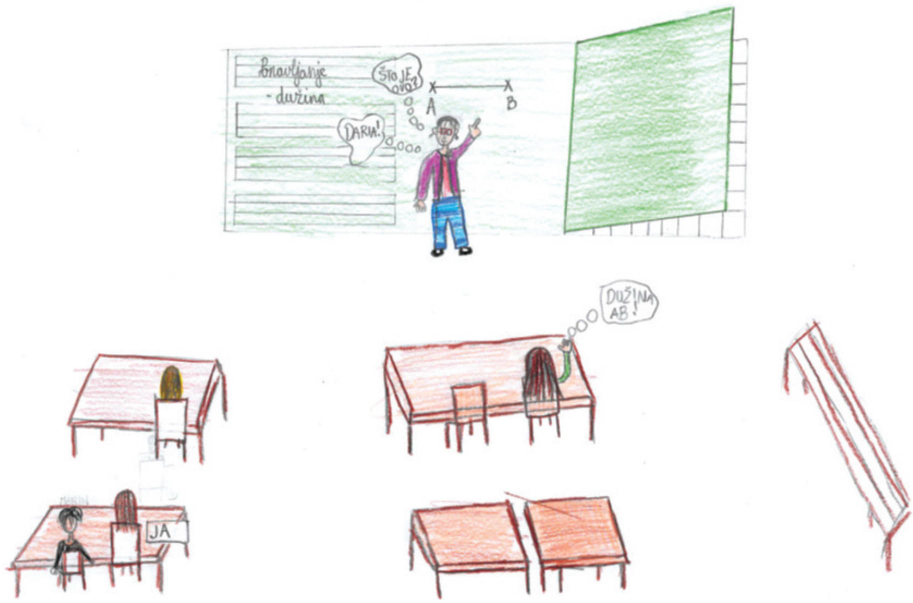
	3a	3b	4a	4c	5a	5c
Number of students	14	16	11	16	16	12

3.2. Research instrument and research process

The research data which consisted of (a) audio data, (b) document review, and (c) a semi-structured interview, were collected in a one-to-one setting between a student and the first author of the paper. The audio data (a) consisted of the students' unprompted verbal reports during the drawing process and prompted verbal reports after the drawing process. For the document review (b), each student was given a blank piece of A4-paper, and was asked to draw their typical geometry lesson. For this purpose, an adaptation of the instrument from the work of Ahtee et al. (2016) and Pehkonen et al. (2011) was used. Before drawing, each student was given a piece of paper and the following instruction was read to them: "Dear __________ , I am Anna and new to your class. I would like to get to know your class better. Draw a picture of your mathematic lesson. The drawing should show what your geometry lessons are like and how you view them. Include your teaching group, the teacher, and the students in your drawing. Use speech bubbles and thought bubbles for conversation and thoughts. Mark the student that represents you in the drawing by writing "ME." Thank you and see you soon! Yours Anna." The students had one school lesson for doing the drawing. No additional instructions were given to the students during the drawing process. Afterwards, as suggested by Kearney and Hyle (2004), a semi-structured interview (c) was conducted with some of the students who were either selected randomly or were selected based on their drawings, in order to gain a more thorough insight into the drawings. During the interview, the students first described what they had drawn (e.g., "Describe your picture to me."), which was followed by specific questions based on the students' description (e.g., "You drew your teacher standing at the blackboard. Is she writing something related to geometry on the board?", "You did not draw all the students. Where are the other students?", "You drew/said that your teacher draws on the board. Does she use a geometric tool for drawing?", "What do the other students do while the teacher is explaining?", "Where are you in your drawing?"). All interviews were documented and served as an additional source of information. Multiple data sources were used in order to assess the consistency of the results and to increase their validity as was suggested by Einarsdóttir (2007) when employing visual research methods.

3.3. Data analysis

Following the data collection, the students' drawings were analyzed. The analysis was carried out in stages utilizing a deductive method, as indicated by Patton (2002). This process involved the following steps: audio data transcription, analysis of drawings with respect to the conceptualization of classroom social climate by Kuzle and Glasnović Gracin (2019, 2021), confirmation of the interpretation by content analysis of the data from the semi-structured interview, and coding of dimensions and respective subdimensions with accompanying scales present in the students' data. Specifically, the first author transcribed the audio data and analysed each of the 85 drawings utilizing a deductively created coding manual that provides descriptions of each component of the classroom social climate model of Kuzle and Glasnović Gracin (2019, 2021). The classroom social climate model consists of three main categories (*Interpersonal Relationship, Personal Growth, and Order*) which are divided into six dimensions, some of which are divided into subdimensions. An example of coding is given in Figure 2. After each drawing was coded within the existing categories and the drawing analysis was completed, descriptive statistics were calculated. Here, a descriptive analysis was made separately for each of the six classes, followed by a comparison of the classes.



Description and coding of the 1st domain "Interpersonal relationship": The teacher is standing in front of the blackboard (Teacher's position in the classroom: in front of the blackboard). She is asking a mathematics-related question and is pointing to the drawing on the board. The following task is illustrated on the blackboard: What is this? (Support by teacher: mathematics-related question). Students are sitting at their desks (Students' position in the classroom: at their desks.) One student is answering the question (Participation: responding,). The involvement of the other three students cannot be identified (Participation: involvement of the students is unidentifiable). The four students are not communicating with each other (Affiliation: no communication with other students). The teacher is standing in front of the blackboard and teaching. The lesson is taught frontally (Working method: teacher-centered instruction (frontal)). The tables are facing the front, arranged in two rows and two columns (Classroom seating arrangement: traditional classroom arrangement).

Description and coding of the 2nd domain "Personal growth": The blackboard shows mathematical content (Goal orientation: the goal of the lesson is clear). Line segments are shown on the board (Teaching materials and tools: 2D-shapes and models).

Description and coding of the 3rd domain "Order": The teacher and the students do not show any behavioral demands in the drawing (Keeping order: unavailable).

Figure 2. Exemplary coding of a Grade 4 student's drawing of a geometry classroom.

4. Results

This section is divided into two parts. The first part focuses on the characteristics of the collective classroom social climate in geometry lessons in grades three to five. The second part focuses on the comparative analysis of the collective classroom social climate in geometry lessons between different school classes.

4.1. The characteristics of the collective classroom climate in geometry lessons

In order to answer the first research question, analysis of the drawings from six different classes was made where each class represents one collective.

Characteristics of the collective classroom climate in geometry lessons (Classes 3.a, 3.b, 4.a, and 4.c)

Here, the results pertaining to classes 3.a, 3.b, 4.a, and 4.c are presented separately. The classes were taught by four different mathematics teachers (see Tables 2, 3 and 4).

Analysis of the collective classroom climate in Class 3.a

The participant-produced drawings showed that in more than half of the students' drawings from Class 3.a the*teacher's position in the classroom* was in front of the blackboard (57.1 %), and in 35.7 % of the drawings, the teacher was sitting at her desk (see Table 2). The position of the teacher could not be determined in a small number of the drawings. In more than half of the drawings, the teacher was making mathematics-related statements (57.1 %), but the teacher asking mathematics-related questions was seen in only a small percentage of the students' drawings (14.3 %). Furthermore, 28.6 % of the drawings did not show teacher support or teacher support could not be identified in them. In the majority of the drawings (85.7 %), students were sitting at their desks. However, in 21.4 % of the drawings, students were shown in front of the blackboard (21.4 %), and in 28.6 % of drawings somewhere else in the classroom. A wide range of student *participation* was shown in the drawings. The number of students who were working on assignments at their desk, listening and/or responding to the teacher's questions were represented in the same percentage of students' drawings (21.4 %). Furthermore, 14.3 % of the drawings show some of the students working on the assignment on the blackboard, taking part in a discussion, or saying something negative about mathematics (e.g., "Ouch!!!", "I don't want to do anything."). In almost half of the drawings (42.9 %), student participation could not be determined. In only a few drawings (7.1 %), were students shown asking for teacher assistance or were shown as being passive. In 85.7 % of the drawings, *student affiliation* was either not shown or not possible to identify. In terms of the *working method*, the teacher standing and teaching at the front of the classroom was depicted in almost half of the drawings (42.9 %), followed by students working individually (14.3 %). Other working forms, namely working with a partner, group work and work/discussion while sitting in a circle were not present in any of the students' drawings. Furthermore, all of the students' drawings reflected a traditional classroom arrangement with tables in rows. Regarding the second category *personal growth*, the students' drawings clearly showed the goal of the lesson. Various teaching materials were depicted in the students' drawings, though line segments, shown in 78.6 % of the drawings, predominated. Students were shown using geometric tools (i.e., rulers) in a very small percentage of drawings (7.1 %) and the teacher was not shown using any geometric tools at all (see Table 3). With respect to the third category *order,* the teacher and the students did not show any behavioral demands in the drawings.

Table 2. Frequency of Interpersonal Relationship.

1. Category: Interpersonal Relationship						
Dimension	Subdimension	Scale	Class 3.a	Class 3.b	Class 4.a	Class 4.c
Verbal and non-verbal communication of the teacher	Teacher's position in the classroom	In front of the blackboard	57.1 %	81.3 %	63.6 %	37.5 %
		Among students	0 %	6.3 %	0 %	6.3 %
		At the desk	35.7 %	12.5 %	27.3 %	37.5 %
		Somewhere else in the classroom	0 %	0 %	0 %	6.3 %
		Unidentifiable	7.1 %	0 %	0 %	0 %
		Unavailable	0 %	0 %	9.1 %	12.5 %
	Support by the teacher	Assistance	0 %	0 %	18.2 %	0 %
		Positive feedback	0 %	6.3 %	9.1 %	0 %
		Negative feedback	0 %	0 %	0 %	0 %
		Mathematics-related question	14.3 %	25.0 %	18.2 %	12.5 %
		Mathematics-related statement	57.1 %	18.8 %	18.2 %	31.3 %
		Observation	0 %	12.5 %	9.1 %	6.3 %
		Non-mathematical comment	0 %	0 %	0 %	12.5 %
		Passive	0 %	0 %	0 %	0 %
		Unavailable/ Unidentifiable	28.6 %	37.5 %	27.3 %	37.5 %
Verbal and non-verbal communication of the students	Students' position in the classroom	Only one or two students shown (at the blackboard/ at their table)	14.3 %	18.8 %	27.3 %	0 %
		At their table	85.7 %	81.3 %	72.7 %	100.0 %
		Next to the teacher	0 %	0 %	0 %	0 %
		In front of the blackboard	21.4 %	25.0 %	18.2 %	18.8 %
		Amongst other students	0 %	0 %	0 %	0 %
		Somewhere else in the classroom	28.6 %	6.3 %	0 %	6.3 %
		Unidentifiable	0 %	0 %	0 %	0 %
		Unavailable	0 %	0 %	0 %	0 %

	Participation	Working on assignments at a table	21.4 %	18.8 %	9.1 %	18.8 %
		Working on an assignment at the blackboard	14.3 %	25.0 %	18.2 %	18.8 %
		Listening	21.4 %	25.0 %	18.2 %	25.0 %
		Responding	21.4 %	18.8 %	18.2 %	6.3 %
		Asking a question	0 %	0 %	9.1 %	0 %
		Asking for assistance	7.1 %	0 %	0 %	0 %
		Review	0 %	0 %	0 %	0 %
		Discussion	14.3 %	43.8 %	9.1 %	0 %
		Positive expression	0 %	12.5 %	0 %	0 %
		Negative expression	14.3 %	0 %	0 %	12.5 %
		Non-mathematical comment	35.7 %	25.0 %	9.1 %	50.0 %
		Passive	7.1 %	12.5 %	9.1 %	25.0 %
		Unidentifiable	42.9 %	25.0 %	36.4 %	68.8 %
		Unavailable	0 %	0 %	0 %	0 %
		Copying from the board	7.1 %	12.5 %	18.2 %	6.3 %
	Affiliation	No communication with other students (while working on an assignment)	14.3 %	18.8 %	9.1 %	12.5 %
		Student-student communication	0 %	0 %	0 %	0 %
		Student-student encouragement	0 %	0 %	0 %	0 %
		Student-student request for help	0 %	0 %	0 %	0 %
		Student-student support	0 %	0 %	0 %	0 %
		Negative comments towards other students	7.1 %	0 %	0 %	6.3 %
		Unidentifiable/ Unavailable	85.7 %	81.3 %	90.9 %	87.5 %
Organization	Working method	Teacher-centered instruction	42.9 %	50.0 %	54.5 %	62.5 %
		Individual work	14.3 %	31.3 %	9.1 %	6.3 %
		Group work	0 %	6.3 %	0 %	0 %
		Working with a partner	0 %	0 %	0 %	0 %
		Work/ discussion while sitting in a circle (half circle)	0 %	0 %	0 %	0 %
		Unidentifiable/ Unavailable	42.9 %	12.5 %	36.4 %	31.3 %
	Classroom seating arrangement	Traditional classroom arrangement	100.0 %	87.5 %	90.9 %	93.8 %
		U-shaped arrangement	0 %	0 %	0 %	0 %
		Mixed arrangement	0 %	0 %	0 %	0 %
		Circle/ Half circle arrangements	0 %	0 %	0 %	0 %
		Group tables	0 %	0 %	0 %	0 %
		Unidentifiable	0 %	12.5 %	9.1 %	0 %
		Unavailable	0 %	0 %	0 %	6.3 %

Table 3. Frequency of Personal Growth.

2. Category: Personal Growth					
Dimension	Scale	Class 3.a	Class 3.b	Class 4.a	Class 4.c
Goal orientation	The goal of the lesson is clear.	100.0 %	93.8 %	81.8 %	93.8 %
	No mathematical content.	0 %	6.3 %	18.2 %	6.3 %
	Teacher identifies/ shows mathematical content	14.3 %	0 %	0 %	0 %
	Students work on their assignment	14.3 %	25.0 %	9.1 %	6.3 %
Teaching materials and tools	2D-shapes	14.3 %	43.8 %	72.7 %	87.5 %
	3D-shapes	7.1 %	18.8 %	27.3 %	6.3 %
	2D-models	0 %	0 %	0 %	0 %
	3D-solids	0 %	6.3 %	0 %	0 %
	Poster	0 %	0 %	0 %	0 %
	Geometric tools (teacher)	0 %	18.8 %	9.1 %	12.5 %
	Geometric tools (students)	7.1 %	0 %	9.1 %	31.3 %
	Unavailable	14.3 %	12.5 %	9.1 %	6.3 %
	Line segment	78.6 %	37.5 %	18.2 %	62.5 %

Analysis of the collective classroom climate in Class 3.b

Regarding the first subdimension, *teacher's position in the classroom*, the data showed that in 81.3 % of the students' drawings, the teacher was in front of the blackboard. As for the second subcategory *support by the teacher*, in 25 % of drawings, the teacher was shown asking mathematics-related questions or making mathematics-related statements (18.8 %) (see Table 2). There was a small percentage of drawings where the teacher was monitoring the students (12.5 %), and giving positive feedback (6.3 %) (e.g., "Correct, Ana.", "Well done."). Although the teacher was drawn in all the students' drawings, in 37.5 % of them, teacher support was not shown or could not be determined. Regarding *students' position in the classroom*, in most of the drawings (81.3 %), the students that were shown were at their desks, only 25 % of the drawings showed students in front of the blackboard. Besides working on the assignments at their desks (18.8 %), various other forms of *participation* were shown, such as working on the assignment on the blackboard, listening, responding, copying from the board, and taking part in a discussion (see Table 2). In 12.5 % of the drawings, students were making positive statements (e.g., "Yes, geometry.", "I love geometry."); there were non-mathematical comments in 25 % of drawings (e.g., "Do you know that guy in Fortnite?"). In a high percentage of the drawings (81.3 %), *student affiliation* was not shown or could not be identified, however, the students were working quietly on their task(s) in 18.8 % of drawings. In half of the drawings, the lesson was taught frontally, and individual work was shown in 31.3 % of drawings. Furthermore, group work was only shown in 6.3 % of drawings, with the working method either not shown or could not be determined in 12.5 % of students' drawings. A traditional classroom arrangement was shown in 87.5 % of the drawings. In 12.5 % of drawings, the seating arrangement could not be identified because only one table was drawn. The data revealed that in 90 % of the drawings, the students perceived the goal of the lesson as being clear, but in a few drawings no mathematical content was shown (see Table 3). In 25 % of drawings, the students were shown working on their assignments. Here, the students drew or mentioned various teaching materials, such as 2D-shapes (43.8 %), line segments (37.5 %), 3D-shapes (18.8 %), and 3D-solids (6.3 %). Nevertheless, geometric tools (i.e., triangles) were only being used by the teacher in 18.8 % of drawings. In regard to the third category, in the majority of students' drawings (81.3 %), neither the teacher nor the students showed any behavioral demands. Still, a number of drawings illustrated the teacher maintaining order (18.8 %) and/or the students (6.3 %).

Analysis of the collective classroom climate in Class 4.a

In 63.6 % of the participant-produced drawings the teacher is standing in front of the blackboard. On a few occasions, the teacher was shown sitting at their desk (27.3 %), which was the second most-frequently coded position. *Support by the teacher* was not shown or could not be determined in 27 % of the drawings. Otherwise, the scales 'mathematics-related statement', 'assistance' and 'mathematics-related question' were shown in the same percentage of the drawings (18.2 %). The teacher was giving positive feedback to students in some drawings (9.1 %), for example, "Well done!". Concerning *students' position in the classroom*, in 72 % of drawings, the students were sitting at their tables, or were shown in front of the board (18 %) in their geometry lessons. Twenty-seven per cent of drawings, showed only one or two students. Besides sitting at their desks working on assignments (9.1 %), various other forms of *participation* were depicted in the drawings, such as doing a task on the blackboard, listening, responding, asking questions, discussion and copying from the board (see Table 2). In 90 % of the drawings, *student affiliation* was either not shown or was not possible to identify. Teacher-centered instruction, where the teacher is at the center of the activity and teaches while the students usually listen and receive information, was coded in more than half of the student drawings, but the working method was either not shown or could not be identified in 36 % of the drawings. Students working individually was shown in only 9.1 % of the drawings. A traditional classroom arrangement with tables in rows was shown in 91 % of drawings, but in some drawings only one table was drawn and the seating arrangement was unidentifiable (9 %). The goal of the lesson was clearly shown in 81.8 % of drawings. The students drew different teaching materials, such as 2D-shapes (72.7 %), 3D-shapes (27.3 %), and line segments (18.2 %) that are used to achieve the lesson goals. Geometric tools (i.e., triangles, dividers and rulers) were being used by the teacher and the students in the same percentage of drawings (9.1 %). Concerning the third category *order*, no behavioral prompts on the part of the teacher or students were shown in any drawing (see Table 4).

Table 4. Frequency of Order.

3. Category: Order					
Dimension	Scale	Class 3.a	Class 3.b	Class 4.a	Class 4.c
Keeping order	Led by students	0 %	6.3 %	0 %	0 %
	Led by teacher	0 %	18.8 %	0 %	6.3 %
	Unavailable	100.0 %	81.3 %	100.0 %	93.8 %

Analysis of the collective classroom climate in Class 4.c

The data showed that the teacher is depicted standing in front of the blackboard and at their desk in the same percentage of students' drawings (37.5 %). The teacher was shown among the students or somewhere else in the classroom in the same percentage of drawings (6.3 %). The teacher was shown making mathematics-related statements and asking mathematics-related questions in 31 % of the drawings and making non-mathematical comments in 12.5 % of the drawings. In terms of s*tudents' position*, in all the drawings, a number of students were shown at their desks, whereas in 18.8 % of drawings, some students were shown in front of the blackboard. Besides sitting and working on tasks (18.8 %), various other forms of *participation* were shown, such as working on the assignment on the blackboard (18.8 %), listening (25 %), responding (6.3 %), and copying from the board (6.3 %). There were non-mathematical comments in half of the drawings (e.g., "The game is great.", "Food.", "Bingo."), in 68 % of drawings the participation of a number of students could not be identified. Some drawings also contained negative expressions (e.g., "I do not need math.", "Boring.", "Ugh.") and depicted passive students (see Table 2). *Student affiliation* was unidentifiable or unavailable in a very high percentage of the drawings (87.5 %) whereas negative comments between students were shown to a much lesser extent (6.3 %). With respect to the *working method*, frontal work was presented in 62 % of drawings, while individual work was shown in only 6.3 % of the drawings. In more than 90 % of drawings, a traditional classroom arrangement was shown, and there was a small percentage of drawings (6.3 %) where no seating arrangement was shown. The data revealed that in more than 90 % of drawings, the students perceived the goal of the lesson as being clear (see Table 3). Regarding *teaching material and tools*, mostly 2D-shapes (87.5 %) were drawn, followed by line segments (62.5 %). Other materials, such as 2D and 3D-models and posters were not shown in any of the drawings. Geometric tools (i.e., rulers, triangles, protractors) were used both by the teacher (12.5 %) and the students (31.3 %). Behavioral demands on the part of the teacher or the students were not depicted in 93.8 % of the drawings. The teacher is shown maintaining order by phrases such as "Vibor, be quiet" or "Silence, go to your place" in only 6.3 % of the drawings.

Analysis of the collective classroom climate in Class 5.a

The results for the collective classroom climate for Class 5.a are given in Tables 5, 6 and 7. With respect to *teacher's position in the classroom*, in 37.5 % of drawings the teacher was shown in front of the blackboard and at their desk in

31 % of drawings. On a few occasions (6.3 %), the teacher was among the students or somewhere else in the classroom. The teacher was asking mathematics-related questions in 43.8 % of drawings, and making mathematics-related statements in 25 % of drawings. Negative feedback and non-mathematical comments were illustrated in 6.3 % of drawings (6.3 %). Regarding*students' position in the classroom,* in 87.5 % of drawings, the students were sitting at their tables; only 12.5 % of the drawings show students in front of the blackboard. Various forms of *participation*, such as listening (43.8 %), responding (25 %), working on assignments at tables (18.8 %) and discussing (18.8 %) are illustrated. In several drawings, some students are making negative expressions (e.g., "Oh, this math.", "It's so boring!"), or making a non-mathematical comment (e.g., "What are you singing?", "I want school to end.") (See Table 5). In the majority of the drawings (87.5 %), *student affiliation* was not shown or was not possible to identify. Students were working quietly on their assignment in 12.5 % of drawings. As represented in Table 6, teacher-centered instruction was the working method most often depicted in the students' drawings (68.8 %); the working method was not shown in 25 % of the drawings. A traditional classroom arrangement with tables in rows was shown in more than 90 % of students' drawings (see Table 6). The goal of the lesson was clearly shown in 69 % of drawings, while the students were working on their assignment in 12.5 % of drawings. With respect to *teaching materials and tools*, the students drew 2D-shapes in most of the drawings (68.8 %), and geometric tools (i.e., triangles, rulers, dividers) were used by the teacher and the students in the same percentage of drawings (12.5 %) (see Table 6). With regard to the third category *order*, in 93.8 % of the drawings, neither the teacher nor the students were giving any instructions regarding behavior, but there is a small percentage of drawings (6.3 %) in which the teacher is instructing students on how to behave (e.g., "Juraj, stop drinking tea. You're not two years old.").

Table 5. Frequency of Interpersonal Relationship.

1. Category: Interpersonal Relationship				
Dimension	Subdimension	Scale	Class 5.a	Class 5.c
Verbal and non-verbal communication of the teacher	Teacher's position in the classroom	In front of the blackboard	37.5 %	100.0 %
		Among students	6.3 %	0 %
		At the desk	31.3 %	0 %
		Somewhere else in the classroom	6.3 %	0 %
		Unidentifiable	0 %	0 %
		Unavailable	18.8 %	0 %
	Support by the teacher	Assistance	0 %	0 %
		Positive feedback	0 %	8.3 %
		Negative feedback	6.3 %	0 %
		Mathematics related question	43.8 %	50.0 %
		Mathematics related statement	25.0 %	33.3 %
		Observation	0 %	0 %
		Non-mathematical comment	6.3 %	0 %
		Passive	0 %	0 %
		Unavailable/Unidentifiable	18.8 %	8.3 %

Verbal and non-verbal communication of the students	Students' position in the classroom	Only one or two students shown (at the blackboard/ at their table)	12.5 %	0 %
		At their table	87.5 %	100.0 %
		Next to the teacher	0 %	0 %
		In front of the blackboard	12.5 %	8.3 %
		Amongst other students	0 %	0 %
		Somewhere else in the classroom	0 %	16.7 %
		Unidentifiable	0 %	0 %
		Unavailable	0 %	0 %
	Participation	Working on assignments at a table	18.8 %	16.7 %
		Working on the assignment at the blackboard	6.3 %	8.3 %
		Listening	43.8 %	25.0 %
		Responding	25.0 %	33.3 %
		Questioning	6.3 %	16.7 %
		Asking for assistance	0 %	8.3 %
		Review	0 %	0 %
		Discussion	18.8 %	41.7 %
		Positive expression	6.3 %	16.7 %
		Negative expression	25.0 %	16.7 %
		Non-mathematical comment	25.0 %	33.3 %
		Passive	18,8 %	33.3 %
		Unidentifiable	18.8 %	33.3 %
		Unavailable	0 %	0 %
		Copying from the board	0 %	0 %
	Affiliation	No communication with other students (while working on the assignments)	12.5 %	8.3 %
		Student-student communication	0 %	0 %
		Student-student encouragement	0 %	0 %
		Student-student help request	0 %	0 %
		Student-student support	0 %	0 %
		Negative comments towards other students	0 %	0 %
		Unidentifiable/Unavailable	87.5 %	91.7 %
Organization	Working method	Teacher-centered instruction	68.8 %	75.0 %
		Individual work	6.3 %	8.3 %
		Group work	0 %	0 %
		Working with a partner	0 %	0 %
		Work/ discussion while sitting in a circle (half circle)	0 %	0 %
		Unidentifiable/ Unavailable	25.0 %	16.7 %
	Classroom seating arrangement	Traditional classroom arrangement	93.8 %	100.0 %
		U-shaped arrangement	0 %	0 %
		Mixed arrangement	0 %	0 %
		Circle/ Half circle arrangements	0 %	0 %
		Group tables	0 %	0 %
		Unidentifiable	6.3 %	0 %
		Unavailable	0 %	0 %

Table 6. Frequency of Personal Growth.

2. Category: Personal Growth			
Dimension	Scale	Class 5.a	Class 5.c
Goal orientation	The goal of the lesson is clear.	68.8 %	100.0 %
	No mathematical content.	31.3 %	0 %
	The teacher identified/ shows the mathematical content	12.5 %	8.3 %
	The students work on their assignment.	12.5 %	8.3 %
Teaching material	2D-shapes	68.8 %	91.7 %
	3D-shapes	6.3 %	25.0 %
	2D-models	0 %	0 %
	3D-solids	6.3 %	0 %
	Poster	0 %	8.3 %
	Geometric tools (teacher)	12.5 %	8.3 %
	Geometric tools (students)	12.5 %	8.3 %
	Unavailable	18.8 %	8.3 %
	Line segment	12.5 %	16.7 %

Analysis of the collective classroom climate in Class 5.c

As illustrated in Table 5, the teacher was shown in front of the blackboard in all the Class 5.c drawings. In half of the drawings, the teacher was asking mathematics-related questions, and in 33 % of drawings the teacher was making mathematics-related statements. Positive feedback from the teacher was shown in 8.3 % of drawings (e.g., "Yes, correct."). Regarding s*tudents' position in the classroom*, there are some students shown at their tables in all the drawings, while students are at the blackboard in less than 10 % of drawings. Besides discussion (40 %), other forms of *participation* were illustrated in the drawings, such as responding and making non-mathematical comments (e.g., "Lovro is very nice.", "I really like Ana.") (see Table 5). In most of the drawings (91.7 %), the *affiliation* of the students was not shown or cannot be identified. As illustrated in Table 6, teacher-centered instruction was shown in 75 % of drawings, while the working method was not shown or could not be determined in 16.7 % of drawings. All of the drawings depict a traditional seating arrangement, and the goal of the lesson was clearly illustrated. The students drew various teaching materials used to achieve the lesson goals, such as 2D-shapes (91.7 %), 3D-shapes (25 %), and line segments (16.7 %). The geometric tools (e.g., triangles, rulers, dividers, protractor) were shown being used by both the teacher and the students (8.3 %). Regarding the third category *order*, in all student drawings, neither the teacher nor the students were giving instructions on how to behave (see Table 7).

Table 7. Frequency of Order.

3. Category: Order			
Dimension	Scale	Class 5.a	Class 5.c
Keeping order	Led by students	0 %	0 %
	Led by teacher	6.3 %	0 %
	Unavailable	93.8 %	100.0 %

A comparative analysis of the collective classroom climate in classes 5.a and 5.c

Since classes 5.a and 5.c were taught by the same mathematics teacher, it was meaningful to compare the characteristics of the classroom climate of these two groups (Tables 5, 6, and 7). The results show many similarities in the drawings of classes 5.a and 5.b. Both present a typically traditional picture of geometry education: teacher-centered instruction, with the teacher asking/stating mathematics-related questions/statements with students mostly at their tables listening to the teacher or responding to her questions, set in the traditional classroom arrangement. Participants from both classes clearly showed that the goal of the lesson was clear. These results may imply that the classroom climate is oriented to mathematics (geometry) in a traditional environment.

However, the results also revealed some discrepancies between the two classes taught by the same teacher. As represented in Table 5, in all Class 5.c drawings, the teacher is in front of the blackboard, while in Class 5.a this is the case in less than 40 % of drawings. Also, fewer children are shown taking part in a discussion in the Class 5.c drawings (41.7 % in 5.a and 18.8 % in 5.c). Students are shown asking for assistance only in a small percentage of Class 5.c drawings (8 %). Although in both classes mostly 2D-shapes were drawn, 3D-shapes were more frequent in 5.c than 5.a. Also, posters were shown only in a small percentage of Class 5.c drawings. Given the small absolute number of participants from classes 5.a and 5.c and the fact that the differences between the two classes are not extreme, this should be further explored.

4.2. A comparative analysis of collective classroom social climate in geometry lessons among classes

Here, we present the results of a comparative analysis which was conducted in order to filter out the similarities and differences in the collective classroom climate of the various classes and was structured around each classroom social climate category. The results are organized according to the classroom social climate categories.

Analysis of the first category: Interpersonal Relationship

The first dimension *verbal and non-verbal communication of the teacher* was defined through its respective subdimensions, *teacher's position in the classroom* and *support by the teacher* with their accompanying scales (see Tables 2 and 5). Regarding the first subdimension *teacher's position in the classroom*, irrespective of the grade level, the teacher was predominantly shown at the blackboard in the participant-produced drawings (see Tables 2 and 5). The teacher was shown as being among the students only in drawings from classes 3.b, 4.c and 5.a (6.3 %). The teacher sitting at their desk was the second most-often coded position in all grades, except in Grade 5.c where this teacher position was not illustrated at all. On a few occasions, in drawings from 4.c and 5.c, the teacher was shown somewhere else in the classroom. In the second subdimension, *support by the teacher*, the scales ‘mathematics-related statement’ and ‘mathematics-related question’ were either the first or the second most-often coded type of support in all the drawings. Only in classes 3.a and 5.c were there any drawings showing students asking for assistance. Non-mathematical comments were present in all classes, ranging between 9.1 % in Class 4.a and 50 % in Class 4.c. *Affiliation* of students was either not shown or not possible to identify in more than 80 % of the drawings. The students did not communicate with other students (while working on the assignments) in some drawings of each grade, with the highest percentage in Grade 3.a (14.3 %). Other aspects of *affiliation* such as student-student encouragement, request for student help, and student support were not illustrated in any of the drawings (see Tables 2 and 5).

Organization, the third dimension, is described by two subdimensions, namely *working method*, and *classroom seating arrangement* with accompanying scales (see Tables 2 and 5). With respect to w*orking method*, teacher-centered instruction with the teacher standing at the front of the classroom and teaching while students take notes was illustrated in almost half of the drawings (42.9 %) with the highest percentage in Class 5.c (75 %). Individual work was the second most-often coded working method in all grades (6.3 %–31.3 %). Group work was present in only one Class 3.b drawing, and working with a partner and working in a (half-)circle were not in any of the drawings. In some drawings the working method was not shown or was not identifiable, with the highest percentage in the drawings from Class 3.a (42.9 %). The working method depicted was related to the *classroom seating arrangement*. Almost 90 % of all the participant-produced drawings reflected a traditional classroom arrangement with tables in rows (see Tables 2 and 5). Other seating arrangements, such as U-shaped, circle/ half circle arrangements and group tables were not shown in any of the students’ drawings. However, in some drawings, either only one table or no tables were drawn and for this reason the classroom seating arrangement could not be identified.

Analysis of the second category: Personal Growth

Personal growth is described through its subdimensions *goal orientation* and *teaching materials and tools*. The students’ data revealed that the goal of the lesson was clear in all Class 3.a and 5.c drawings, and in other classes the goal of the lesson was clear in at least 60 % of drawings. In only a small percentage of the drawings from classes 3.b, 4.a, 4.c and 5.a was no mathematical content shown. According to data, the teacher is indicating mathematical content only in the drawings from classes 3.a, 5.a and 5.c with the highest percentage in 3.a (14.3 %) (see Tables 3 and 6). The participant-produced drawings showed some students in each class working on their assignments, with the highest percentage in Class 3.b (25 %). In order to achieve the lesson goals, different teaching materials and tools specific to geometry were illustrated in the students’ drawings. For instance, 2D-shapes and line segments were drawn most often, irrespective of the grade level. Whilst 2D-shapes were most common in Class 5.c drawings (91.7 %), there were most line segments in Class 3.a drawings (78.6 %). 3D-shapes were also depicted in all grades but in a smaller percentage (6.3 %-27.3 %). One student from Class 3.b and one from Class 5.a drew 3D-solids, but 2D-models were not illustrated in any of the students' drawings. Geometric tools were used minimally by the teacher (8.3 %–18.8 %) in all classes, except for in 3.a where this was not illustrated at all. Geometric tools were drawn the most frequently by Class 4.c students (31.3 %).

Analysis of the third category: Order

With regard to the third category, *order*, in almost all of the drawings (81.3 %) regardless of the grade level, behavioral incentives on both the part of the teacher and students were not illustrated. Behavioral incentives on the part of the teacher were present in the drawings of classes 3.b, 4.c and 5.a, with the highest percentage in 3.b (18.8 %). Behavioral prompts are only illustrated in drawings by students from Class 3.b (6.3 %) (see Tables 4 and 7).

5. Discussion and conclusions

In the last section, the key aspects of collective classroom social climate in geometry lessons in Croatia are discussed. The limitations of the study are considered and some possible future research directions are provided.

5.1. Collective classroom social climate in geometry lessons in the Republic of Croatia

The main goal of this study was to analyze and compare the characteristics of the classroom social climate of different collectives (each class represents one collective) for which six different classes were selected by using an instrument of Kuzle and Glasnović Gracin (2021). In this study, as in previous studies (Kuzle, 2023; Kuzle & Glasnović Gracin, 2019, 2021), the instrument has proven to be viable, with all 74 dimensions and subdimensions being illustrated in the students' drawings.

The results showed that in the majority of the students' drawings the teacher is shown standing at the front of the classroom, but there were differences between the grades regarding the other positions of the teacher. The teacher was drawn among the students in classes 3.b, 4.c and 5.a only, and the teacher giving assistance was only depicted in Class 4.a. Furthermore, positive feedback from the teacher was illustrated in only a few drawings from classes 3.b, 4.a, 4.c and 5.c; negative feedback was only illustrated in Class 5.a students' drawings. Students at their desks was the most commonly drawn student position in all the classes overall, but this position was most frequently shown in the drawings from Class 3.a. Also, there were differences in the representation of other student positions. The highest proportion of students shown in front of the blackboard was in 3.b. Listening was identified most in Class 5.a drawings, and asking questions was seen most in 5.c drawings. Teacher-centered instruction was illustrated in all classes but with the highest percentage in 5.c. (75 %). In all classes, the second most-often coded working method was individual work; group work was only present in one drawing from Class 3.b. In all of the drawings by classes 3.a and 5.c the goal of the lesson was clear, and there was only a small percentage of drawings overall where no mathematical content was shown. Some students in each grade were shown working on their assignments, with the highest percentage in Class 3.b (25 %). As previously stated, different teaching materials and tools were illustrated in the students' drawings. While 2D-shapes were shown most often in Class 5.c, there were most line segments in Class 3.a drawings. Geometric tools were drawn being used by teachers in all classes except for 3.a where no geometric tools were illustrated at all. The highest number of drawings showing students using geometric tools is from Class 4.c. Regarding order, the third category, the teacher was shown keeping order in a small number of drawings in classes 3.b, 4.c and 5.a, with the highest percentage in 3.b. Order was shown being kept by students only in drawings by Class 3.b students.

In this study, students showed their geometry lessons being taught frontally with the teacher standing at the blackboard and teaching (more than 60 % of drawings). Additionally, the participants presented the teacher making mathematics-related statements (e.g., "The triangle is . . . ") or asking mathematics-related questions (e.g., "What is the name of the 2D-shape which is drawn on the blackboard?", "What is a line segment?") while standing in front of the blackboard. The teacher giving assistance was only shown in a few Class 4.a drawings. According to Moos and Moos (1978), teacher assistance, as one of aspect of the classroom social climate, is important because it can help keep students' attention in the classroom. In more than 85 % of the drawings students were sitting at their desks in their geometry lesson. The students' data revealed a broad spectrum of *participation* in geometry lessons, such as working on assignments, listening to the teacher, copying from the board, and asking a question. As a result, any generalizations related to this component of classroom social climate are precluded. Social relationships within the classroom were more visible between students and teachers than in communication between the students. In 89 % of students' drawings, the goal of the lesson was clearly shown. These findings correspond to the findings of Kuzle and Glasnović Gracin (2019), and Kuzle (2023). In addition, in a very high percentage of drawings (94 %) a traditional classroom seating arrangement was shown, as another component of traditional teaching methods. Traditional teaching methods, according to Swan (2006), can have a negative effect on students' attitudes toward mathematics. In order for geometry teaching to be successful, it is necessary to use different teaching tools and materials. Using concrete manipulatives in geometry lessons can also affect the development of fine motor skills (Jones, 2002). The data in this study also revealed that different teaching tools (e.g., triangles, rulers, dividers, protractor) and materials are used in geometry lessons but with a predominance of 2D-shapes and line segments. According to Bobis et al. (2011), using various materials and tools enables students to explore various mathematical concepts and processes by manipulating them. Furthermore, in more than 94 % of drawings no discipline problems were shown. Given that there were very few instances of negative classroom characteristics, the classrooms established a predominantly positive emotional classroom climate (Kuzle, 2023) which is a further positive indicator of classroom social climate.

The results show a tendency for geometry lessons to be taught in a rather traditional environment, but there are different collective classroom climates in different classes. This finding is understandable given that each of the grade 3 and 4 classes were taught different by teachers. The comparison of results in classes

5.a and 5.c which were taught by the same mathematics teacher, shows a broadly similar classroom social climate. However, some differences suggest that not only the teacher but also the students contribute to the classroom social climate and its specific characteristics. This is not surprising given that students learn and live in the classroom almost every day; each student is a part of the classroom, and he or she influences the creation of a classroom social climate (Evans et al., 2009; Moos & Moos, 1978; Trickett & Moos, 1973). These findings are in accordance with Kantorová (2009) who stated that every collective has a different classroom climate which depends on the specific situations of each collective. Classroom social climate therefore represents an essential element of teaching that should not be underestimated for both the students and the teachers (Meyer, 2016).

5.2. Limitations of the Study and Future Research Directions

This research was an exploratory qualitative study using purposive sampling (Patton, 2002). In this study a sample of six classes with a total of 85 participants from one school was used. However, as only one school with six different classes and five different mathematics teachers participated in the research, the results may be limited and for that reason may not be widely generalized. Furthermore, it cannot be supposed that a comprehensive and objective picture of the classroom social climate was presented through the drawings as there were some drawbacks, for example, some students did not like drawing, some had difficulties drawing, and some had limited expressiveness of some aspects using drawings, as was also reported in earlier studies using drawings (e.g., Ahtee et al., 2016; Kuzle, 2023; Kuzle & Glasnović Gracin, 2018, 2019; Pehkonen et al, 2011). Further, the data sources offered only one perspective, that of the students. According to Maturana (1988), the classroom social climate can only be understood when three perspectives are taken into consideration: the perspectives of the students, the teacher, and the researcher.

This limitation suggests a possible next step in the research process, such as expanding the study methodologically and refining the tools used. Future research could use a much larger data sample from a wider variety of settings (e.g., different schools from all parts of Croatia) in order to obtain a broader picture of the characteristics of collective classroom social climate. Drawings from entire classes from different grades and schools may also give a more complete picture of how primary school students conceptualize classroom social climate in geometry lessons. This would also allow us to make comparisons between different grades and schools. Furthemore, due to the study design, we were only able to make direct conclusions about students' perceptions of their geometry lessons pertaining to classroom social climate. Hence, in future studies, additional data sources should be used, such as observations of geometry lessons. In that manner, a less biased and more thorough description of the collective classroom social climate could be attained. Lastly, in this study we presented the characteristics of the collective classroom social climate in geometry lessons and these findings might be limited to the characteristics of this area of mathematics. Consequently, future research may include investigating the characteristics of collective classroom social climate in other mathematical areas (e.g., arithmetic), in order to gain a more comprehensive understanding of the characteristics of collective classroom social climate in mathematics lessons as a whole. Despite these drawbacks, students' drawings and the processes by which they are made open up a new way of obtaining insight into the classroom social climate, especially in mathematics education research.

This research, the main purpose of which was to investigate the characteristics of the collective classroom social climate, was the first of this kind to be conducted in the Republic of Croatia in the context of primary school geometry. It provided insights into the characteristics of collective classroom social climate in geometry teaching, and with it, new information and additional knowledge about geometry teaching in primary education in the Republic of Croatia, which may influence future changes to the curriculum.

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Kolektivna razredna socijalna klima u hrvatskoj nastavi geometrije: Analiza učeničkih crteža

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Sažetak. Učionica predstavlja društveni kontekst u kojem učenici svakodnevno djeluju i uče. Tijekom vremena svaka učionica razvija specifičnu društvenu klimu koja posjeduje različite karakteristike. Budući da se razredna klima odnosi na specifične kurikularne i pedagoške komponente, ona može biti različita u različitim nastavnim predmetima kao i između tema (npr. unutar matematike). Cilj ovog rada bio je ispitati kolektivnu socijalnu klimu na satu geometrije prema učeničkim crtežima. Razredna socijalna klima analizirana je na temelju triju kategorija: interpersonalnih odnosa, osobnog razvoja i održavanja reda, i to u u šest razreda iste osnovne škole (po dva razredna odjela od 3. do 5. razreda). Rezultati su pokazali neke sličnosti i razlike među ispitanim razredima. Razlike su se odnosile na aktivnosti učitelja (učitelj daje matematičke izjave ili postavlja matematička pitanja), učeničkoj aktivnosti (učenici raspravljaju ili pasivno slušaju) i nastavnim materijalima i alatima specifičnima za nastavu geometrije (npr., geometrijski likovi i modeli, geometrijska tijela i modeli, geometrijski alati). Rezultati su pokazali da je većina ispitanih razrednih odjeljenja prikazala nastavu geometrije kao frontalnu nastavu s učiteljicom ispred ploče te s učenicima u klupama u tradicionalnom rasporedu sjedenja. Cilj nastave bio je pritom jasno prikazan na gotovo svim crtežima. U širem smislu, sličnosti u razrednoj klimi na satovima geometrije prema razrednim odjeljenjima i razrednim razinama također mogu biti indikativne za aspekte školske klime.

Ključne riječi: socijalna klima, kolektivna klima, geometrija, osnovna škola, razredna nastava

Connecting Statistics with Spatial Abilities

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Abstract. Our earlier research showed that the teaching of spatial geometry has been significantly reduced in the Hungarian education system in recent decades. Similar tendencies are well known also internationally. Thus, one of the key elements that could be useful in developing spatial intelligence was reduced in mathematics education. This is problematic because many research proved that the development of spatial abilities has a positive effect on the completion of science and STEM subjects, so it would help to reach the coveted reduction of drop-out rates. In addition, many studies argue that spatial abilities should also play a greater role in teaching of e.g. geography, GIS, sociology, medicine, business sciences and marketing. On the other hand, research proves that the improvement of space intelligence is of importance in therapies of dyslexia and dyscalculia.

In contrast to all these, there is a trend that is contrary to the above: increased expansion of statistics content in mathematics teaching can be observed. Taking advantage of this fact, the presentation focuses to look at how statistical tasks could be used to develop spatial thinking. We make suggestions on the types of this kind of tasks and analyse how they would be capable of developing spatial intelligence factors, especially those of mental rotation and visualization.

Keywords: teaching statistics, spatial intelligence, visualization, mental rotation, reducing drop-out rates

1. Introduction

In this paper, we ask what non-standard methods might be suitable for developing spatial skills in secondary school mathematics education? Our investigations are based on the Hungarian context, with the belief that similar solutions can be put into practice abroad.

The paper is structured as follows: after a general introduction, we review the history of curricular change in Hungary, with a special focus on statistics. Then we briefly summarize why we consider the development of spatial intelligence in public education to be of particular importance and discuss the related theories and development opportunities. In the final chapter of the paper, we propose a program that combines the development of spatial intelligence with the methods of statistics and provides an opportunity to develop both areas simultaneously.

We begin with the truism, but very important truth, that we live in a fast-paced world that is accelerating. Progress in every technological, industrial, and economic field is exponential, and the amount of information produced by mankind is increasing at a frantic pace. It is estimated that in 2025, roughly a week will generate as much data as was generated in the whole of 2013 (4,3 zettabytes), when it was already more than 850 times the amount of data stored in all the books ever written (5 exabytes) (Arbia, 2023). Not only is the amount of data constantly increasing, but the need to access and therefore process it has also increased. The results of today's digital revolution are knocking at the door, perhaps just refer to the news tsunami around Artificial Intelligence, which has made the subject a hot topic in recent months. Our lives have changed fundamentally over the last decades and of course schools must follow suit. Obviously, we recognize that it cannot be taught the same and in the same way as it was e.g. in the 2000s. Education must obviously change both in form and in content. We feel that the words of Papert and Caperton are a pertinent quote: "Digital technology in the workplace requires a new definition of 'basic skills". The transformation of work requires much more than a mastery of a fixed curriculum inherited from past centuries. Success in the slowly changing worlds of past centuries came from being able to do well what you were taught to do. Success in the rapidly changing world of the future depends on being able to do well what you were not taught to do." (Papert & Caperton, 1999) This need was already expressed in Hungary in 2009 by Neményi and Somfai (2009), when in their study they stated that "we have to adapt to the expectations of today's world. There is a need for more practical mathematics (economic problems, graphs, statistics, probability, etc.), which is of interest to the majority of students and is therefore a major motivating factor. This means that we need to prepare for a change of emphasis in our teaching." The presentation of statistical content is particularly justified by the fact that a large part of our information sources are infographics, which make the data themselves visually interpretable, i.e., help us to "see and be seen". In simple terms, infographics are a combination of information and graphics, i.e., data and message, linked by visuality. The visual effects take the form of graphic elements (shape, color, illusions (spatial effects, shading)). The form associated with the information (graph, chart, data series,...) is not entirely optional, it must reflect the nature of the information. Interpreting or producing such data visualization infographics is unthinkable without basic statistical knowledge.

With all this in mind, we feel that it is appropriate to review the curriculum regularly and to give a greater role to statistics in it, but perhaps even more important than setting the curriculum is that schools develop basic skills. This is the only way to ensure that future generations can successfully adapt to changed circumstances. At this point, the author starts to iterate his favorite theme, since his experience and conviction in technical higher education for almost three decades has shown that it is important to develop spatial skills and to emphasize visuality, at least in the teaching of mathematics (with the withering away of the subject of Drawing/Visual Culture), for the widest range of students. As the traditional medium of this, descriptive geometry, and more broadly spatial geometry and geometry itself, has been relegated to the background (Bölcskei, 2022), a new medium is needed to achieve the desired goal.

2. Curriculum changes in Hungary and their impact

In this chapter, we briefly review the changes in curricula (mainly in secondary schools) in Hungary over the last 45 years. Starting with the more recent times, it can be said that the regulation of education is implemented through a complex system of several stages, the main elements of which are the National Core Curriculum (NCC), the framework curricula, the local curricula, the approved textbooks and the examination system. Prior to the introduction of the National Curriculum, which defines and guides the whole system, there was a centralized, uniform central curriculum inherited from the socialist system and introduced in 1978. The National Curriculum itself was introduced in 1995 and has been revised four times since then: in 2003, 2007, 2012 and 2020. The last two of these revisions were major. The first three NCCs sought to break down a centralized structure and gave local curricula the opportunity to innovatively fill in the content of the NCC framework (areas of learning and lesson ratios, key competence areas, development areas, educational objectives). The last two versions are a more centralized system, where the framework is more precisely defined by the framework curricula and where, in essence, local curricula cannot deviate from the framework. The framework curricula are therefore the most important of the enacted regulators today, and changes in the discipline of statistics are therefore dealt with in terms of framework curricula.

The period between 1978 and 1995 was briefly characterized by the fact that statistics was not taught within the framework of mathematics. Probability and the related combinatorics were also given a marginal role, but neither the curriculum nor the final secondary school examination included any statistical exercises.

The 1995 change in the teaching of statistics has not yet had a significant impact appearing only as headings in the curricula developed up to grade 10 at the time, such as data collection, the use of a statistical yearbook, data classification, analysis and visualization, without mentioning specific mathematical and statistical concepts.

The 2000 Framework Curriculum gives much more attention to statistics, emphasizes its usefulness and highlights its importance in the social sciences. Descriptive statistics and the representation of data sets (pie charts, bar charts) are mentioned, as are the mode, median and standard deviation for characterizing data sets (Grade 9). In Grade 11, the task is to process statistical data, to put them into the context of everyday problems and to practice sampling. In Grade 12, the practical role of descriptive statistics is the topic, using a computer. The concepts of classification and range are also introduced, as well as public opinion research and quality control as an area of application.

The 2003 framework curriculum is essentially the same as above, only slightly reduced for Grade 12, as the areas of use are no longer included.

The next NCC itself was published in 2007 but the corresponding framework curricula was published one year later, just in 2008. Its formulation and structure differ from the previous ones, giving different regulations to the variety of schools. As an interesting example we mention that in the 6^{th} grade statistics and functions form one teaching unit. In the higher classes the content also differs slightly from the previous ones: in the 9th grade, the band and line diagrams appear in data visualization in addition to the bar and pie charts. For data collection, a statistics pocketbook is suggested, the concept of the mean has become an expected knowledge, but standard deviation is not yet an exercise here, only in grade 10, as are range and mean absolute deviation. However, in the 11th and 12th grade curricula, the emphasis is on probability, in which statistics is only mentioned in this context.

The next framework curriculum for 2012, as mentioned above, is binding for schools. It does not define the curriculum by grades but by grade pairs (9–10 and 11–12). In grades 9–10, the emphasis is on reading tables and diagrams (line, bar, pie charts), and the curriculum also covers the mean, the mode and the median. In the upper two grades, statistical sampling is added and, as the concept of range, again. Public opinion research and quality control are again included in the curriculum and reference is made to the use of computers. The concept of variance is only introduced in the systematic summary at the very end.

The latest and still valid curriculum is the 2020 curriculum. It also defines the knowledge for grades 9–10 and 11–12, but more knowledge than before is included with more clearly defined objectives and tasks. In grades 9–10, the tasks are: designing and carrying out data collection, mean, mode, median (mean values – these are already introduced in grades 7–8 in this new system). In addition, to draw conclusions from them, use, suitability, inter-conversion of bar and pie charts, recognition of graphical manipulations, all these based on statistics on sport, transport and school results. In Years 11–12, however, the above is complemented by the concepts of quartile, range, standard deviation, minimum, maximum, outlier, and using box-plot diagrams, analyzing data sets and drawing conclusions, with applications of voting statistics and analyzing databases from the Hungarian Central Statistical Office. The Simpson's paradox, which is a contradiction, whereby a finding for a group can be reversed if the group is further decomposed, is also part of the course.

As can be seen, statistical topics have been more or less present in school education over the last 20 years or so, as the mean values, types of charts or measures of variance and areas of application have varied constantly. It is certain that the 2020 curriculum contains the most knowledge and is the most complex of all of them.

The Table 1 shows the changes in the number of math lessons per grade.

By way of clarification, the 2003 curriculum also includes 96 hours for art classes in the final year. The 2008 system is very complicated, since it distinguishes between the following subjects in the higher-level gifted curricula: Hungarian, history, mathematics, physics, biology and chemistry. The table shows the number of lessons in a typical humanity and a typical science module. It is also important to note that for the last two curricula, not only the requirements but also the number of hours are specified for grade pairs only.

Table 1. Number of lessons in Math per year.

Curricula	Grade 9	Grade 10	Grade 11	Grade 12
2000.	111	111	111	128
2003.	111	111	111	128 or 96
2008.	92.5 or 111	92.5 or 111	92.5 or 111	80 or 96
2012.	204		186	
2020.	204		186	

Overall, the table shows that the cumulated number of mathematics lessons has decreased from 461 to 390 over the last 20 years, which means a reduction of about 15 %.

The number of hours devoted to statistics is not easy to extract from the curricula, because it is often given in combination with probability (sometimes with other subjects). What is known is that in 2008, probability and statistics together accounted for 22 contact hours over the four years. In 2012, the same number will be 30 (10 hours in 9–10, 20 hours in 11–12), while by 2020, 10 hours will be dedicated to statistics alone in Grade 9–10 and 12 hours to statistics alone in 11–12. This is an eye-opening sign of an expansion.

It should also be noted that the National Curriculum and the Framework Curricula also assign tasks related to the application of statistics in other areas of education, such as Computer Science (where the task is to manage large data systems and visualize data), Earth and Environment (where the task is to interpret economic or geographical statistical data, e.g. age structure of a population), History and Social Studies (where the task is to interpret population, geography of settlements based on graphs and diagrams).

This paper does not deal with the situation in specialized secondary schools where more in-depth statistical studies are required in business studies, economics, tourism, or marketing.

In the light of the above, we see an opportunity to improve the declining geometric competences by increasing statistical knowledge and hours.

More specifically, we see an opportunity to combine tasks related to the visualization of statistical data, data visualization and the development of spatial intelligence.

Let me briefly digress here and systematize the knowledge on data visualization. Ritchie (n.d.) considers data visualization as a special type of infographics and states that the most important goal is to display data accurately. This is more difficult than it may first appear, as it is not only necessary to make the data understandable, but also to make it easier to recall and analyse. Good data visualization not only makes the data visually appealing, but also tells a story. It does this in an aesthetic way that is pleasing to the eye, but the focus remains on the data. As an infographic genre, data visualization is always objective and never manipulative (Ritchie et al., 2012).

In another approach (Manovich, 2020), data visualization is nothing more than a mapping between data and its visual representation, or more precisely a mapping between two modalities of the brain, namely the mathematical and the visual. According to Manovich, the goal of data visualization is to make large amounts of data transparent to others. His focus is therefore on the communication of information, but he also has a secondary goal. It can also help statisticians if, after data collection and during the data processing phase, it can show patterns and relationships between variables by making the data transparent, and then help to select the appropriate statistical method.

To summarize, data visualization is a method of visualizing relationships between numbers (data) in order to illustrate, for example, trends in processes, identify patterns, find outliers, etc.

The Big Data phenomenon also presents a new challenge, as it requires new methods of data visualization in addition to the traditional ones, thus facilitating data processing and data mining.

3. Overview of the concept of spatial intelligence and how it can be developed

It is worth clarifying what exactly we mean by the concept of spatial abilities. Two common and established definitions are mentioned. According to McGee (1979), it is the ability to mentally manipulate, rotate, twist, or invert pictorially. presented stimulus objects. A common definition in the Hungarian literature (Séra et al., 2002) focuses on spatial problem solving: “Visual-spatial ability is defined as the ability to perceive two- and three-dimensional shapes and to use the perceived information to understand objects and relationships and to solve problems.”

The first observation about spatial intelligence dates back to 1880’s, when Galton (1883) noted that it exists separately from general intelligence. In relation to research on spatial vision, it is worth highlighting the work of Thurstone (1947), who in his model of intelligence distinguished between the components of verbal comprehension, verbal fluency, number, perceptual speed, inductive reasoning, and spatial visualization and made it possible to measure these using the Primary Mental Abilities Test he developed. This is important because he is the first to present spatial visualization as a separate factor. To measure spatial abilities, he created tasks involving the mental rotation of different planar shapes (cards, flags).

As far as contemporary theories of intelligence are concerned, most of them accept the existence of a general intelligence factor, a single cognitive ability. Intelligence is conceptualised in a hierarchical model, where one of the primary factors associated with general intelligence is the spatial-mechanical factor and the other is the verbal-learning factor. These two main components – verbal and spatial – seem to represent the fundamental dichotomy in human cognition.

In this regard, it is interesting to note that while verbal-linguistic impairments are easily detected in early school age and for which a number of tests are available, the test for spatial impairments is new and not particularly widespread (Cornoldi et al., 2003), a fact that faithfully reflects the greater focus on verbal skills and the underestimation of spatial skills.

In contrast, Arnheim (1980) argues that thinking is primarily visual thinking. He sees no evidence that logical thinking takes priority over visual thinking in the functioning of the brain. He claims that this devaluation of visuality is a consequence of Descartes' teachings.

Finally, the traditional approach fails to take into account the many research results of recent decades that have shown the usefulness of spatial intelligence in the learning process in a number of disciplines.

To start with, spatial skills are a demonstrable way to help you learn mathematics. This is a seemingly natural and indeed proven fact by many researchers. Stranger still, perhaps, is that spatial vision helps in many other areas: spatial skills have been shown to be successful in the treatment of dyslexia and dyscalculia (Maier, 1996). A likely explanation for this is the positive interaction and interconnection of intelligence factors and skills. This publication also contains references to the fact that the development of spatial vision may be associated with success in biology, chemistry and physical education. The literature is rich in findings that show an undeniable link between spatial intelligence and successful completion of STEM (Science, Technology, Engineering and Mathematics) subjects. Another example from the field of science points to a link between spatial skills and geography. For example, Sanchez's (2012) studies have shown that a better understanding of plate tectonics is also helped by improved spatial skills – or, as a more recent example we mention GIS (geoinformation systems), a complex geographic information system with many spatial aspects. In addition, the development of a spatial perspective is also beneficial for more distant disciplines. At one conference, the author was contacted by physicians who thought that they could benefit from a program to improve spatial perception. This would be useful, for example, during surgery, to get a preliminary idea of the surgical area. Logan (2012) argues that spatial thinking should play a greater role in sociology. Zwartjes et al. (2017) argue that spatial thinking is also relevant in business and marketing.

Newcombe (2013) cites research showing that students with more advanced spatial skills learn more easily from visualizations. Visuals are not only useful for illustrating results (e.g. Mendeleev's periodic table, which makes sense of the similar chemical properties of groups of elements; or the various maps, which illustrate geography, economics, history, etc. at the same time), but also for research. Scientists often draw while making observations or when trying to develop ideas in discussions with other scientists. Newcombe identifies five reasons why active sketching is a good idea at any level: it engages participants, deepens understanding, requires reasoning, forces clarity of ideas, and supports communication within working groups.

In a study by Hall et al. (2021), it has been investigated whether it is necessary to map cognitive abilities across disciplines to develop good visualization strategies, or whether there is a common key that fits all disciplines? They started from the premise that the spatial ability structure represents the way people are able to visualize data – i.e. to process data on the one hand and to represent the relationships between them on the other. In their article, they show that the level of spatial ability varies according to the professions they study (chemist, computer scientist, teacher). Therefore, it is not surprising that there is a correlation between the professions and the results of visual ability tests. This implies that there is no "philosopher's stone" in visual education, and that subject-specific development is needed. Their research also shows that visuality is used in all subject areas, although obviously to different degrees.

Within the spatial perspective, it is important to examine which abilities appear as independent factors, i.e. well separable from other abilities. Different theories show a wide variation in the number of factors and their content. In the earliest systems, only two factors were distinguished: perception and visualization. McGee (1979) already divided the factors into two groups: spatial orientation and spatial visualization. In the former, the object is fixed and we change our position, in the latter the observer is fixed and the object moves. In his system, visualization is further differentiated into mental rotation and mental transformation (McGee, 1979), so that he ends up working with three factors. We note that since the authors did not define each ability element precisely (they were generally interpreted in relation to the test they used), many misunderstandings and misinterpretations have been made.

The most widely accepted system today (Maier, 1996) distinguishes five components, each of which can be developed separately. The names of the components and the contents they denote, in a narrow and wider sense, are as follows:

Spatial perception: narrowly, the ability to recognize vertical and horizontal directions independently of the observer, even in the presence of misleading visual stimuli. This is a static mental process, i.e. the position of the observer may change, but the internal context of the object remains unchanged. The focus is therefore on the recognition of perpendicular directions (and therefore perpendicular spatial elements). In a broader sense, it is the reception of visual stimuli, perception itself and its processing.

Spatial visualization: the ability to visualize an object as its parts move or shift relative to each other. It is a dynamic mental process, as it involves movement and the position of the observer is neutral. An example might be the mental unfolding of a body (finding the relation between the polyhedron and its development). In a broader sense, it is used to refer to tasks involving complex manipulations involving several steps or tasks that do not belong to any other category.

Mental rotations: the ability to rapidly mentally rotate two- and three-dimensional shapes. Dynamic mental process, the position of the observer is irrelevant.

Spatial orientation: a dynamic process whereby the observer can orient himself correctly in a spatial situation, either real or imaginary. In this factor, the position of the observer is the key, the position of the objects does not change.

Spatial relations: this is another static mental process, the essence of which is to understand the spatial relationships inside a figure or between its parts. An example is the relationship between a body and its different views in different directions. The position of the observer is again significant.

There are no sharp boundaries between the different categories, as can be seen from the summary table below.

Table 2. Factors of spatial intelligence according to the position of the observer and the nature of the mental process (Source: (Maier,1996)).

<table>
<tr><th>spatial position of test person</th><th>dynamic mental processes</th><th colspan="2">static mental processes</th></tr>
<tr><td rowspan="2">outside position</td><td>visualization</td><td colspan="2">spatial relations</td></tr>
<tr><td colspan="2">mental rotation</td><td rowspan="2">spatial perception</td></tr>
<tr><td>inside position</td><td colspan="2">spatial orientation</td></tr>
</table>

Without going into the rich literature on the various tests of spatial perception, we merely point out that mental rotation is also measured by means of plane shapes alone (e.g., flags, plane figures, etc., must be marked from among identical elements that are only rotated), and that visualization tests include a variety of tasks: putting together a shape from elements; recognizing a shape as part of another, imagining the shape of a plane section, folding paper, etc.

No matter how we choose our interpretive framework and whatever factors we adopt, we have the opportunity to develop them. Maier (1996) himself offers several ways to do this. In Uttal et al. (2013), the authors compare the results of no fewer than 217 studies, showing that training can improve spatial skills with different kind of training programs. With them improvements in spatial skills test scores can be achieved, and these improvements do not disappear even after a longer period of time.

As mentioned earlier, the development of visual-spatial skills is a neglected area in public education, and in many cases remains hidden. An exception in this respect is higher education in engineering (and STEM fields more broadly), where poor spatial intelligence appears to be a major cause of drop-out. It is therefore no coincidence that the measurement of spatial intelligence is widespread in engineering and architecture universities both in Hungary and abroad. Research clearly demonstrates that the spatial perception of entering students is weak, but it can be improved and developed to different extents, even at this late age (Bölcskei et al., 2012; Kovács et. al., 2014; Sorby et al., 2018). The short training opportunities do not allow for the full catch-up, but selected areas can be developed in a targeted way. To avoid fire-fighting, it is also important to set up appropriate development programs at primary and secondary school levels (Babály et al., 2016; Kárpáti et al., 2014), and this paper serves this purpose.

4. Data visualization to improve spatial vision

The number of tools available for data visualization is growing. Although the current curriculum only requires pie charts, line charts and box-plot diagrams, be aware that there are many non-standard methods of representation, such as waterfall, radar, hierarchy (tree map), map, beanplot, Sankey diagrams; map; data bars; 3D surface, 3D point cloud, bubble diagrams, etc. If you want to represent a more complex system, and one quantity is a function of several others, you either follow a cross-sectional representation, or try to use higher dimensional representations (surface, point cloud) or encode the third and further dimensions with size and color (bubble diagram). Among the representations based on icons, one should mention the representation called Chernoff faces. This can represent multidimensional data and recognize patterns by varying the shape and size of parts of the face (eyes, ears, nose, mouth) up to 18 variables. For more information see e.g. Chernoff (1973), Kelecsényi (2019) or `gapminder.org`.

All the techniques in this paper allow the development of a spatial intelligence factor, as we shall see. In terms of their purpose, visualizations can either show relationships between quantities or help to understand the time course of phenomena, even for complex processes (e.g. Sankey diagrams).

There are five stages in the processing levels of data visualization.

Given a visualization, for example a graph, the first level of interpretation is a simple readout of the data – what was the average income in the US in 1931, what was the average temperature on 13 April in Osijek, what was the population of Croatia in 2010, etc. This is the level of the primary school.

At the second level of interpretation, the observer can find connections between the data based on the diagram – how long the car is parked, when it is moving; what is the average of the data plotted, what is the range, how many times one quantity is greater than the other, etc. This type of task can be found in education at the end of primary school and the beginning of secondary school.

At the third level, the recipient can also make predictions, read trends from a graph – e.g. what will the Earth's population be in 2030, what is the expected GDP, etc. These tasks should be done in secondary school.

At the fourth level, the mere recipient becomes the creator of the graph: in a given interpretative framework, he or she first manually translates the data into a picture (e.g. often by making a pie chart in secondary school) and later does this automatically, using a program (computer, even in secondary school).

At level five, the ability to choose the most appropriate representation for a set of data, i.e. to produce simple or complex infographics, is something that students may only be expected to do at higher education level (Kelecsényi, 2019).

In summary, from a data visualization perspective, there are two main roles: the creator, who creates the data; and the receiver, who interprets the data and makes decisions based on it. In both roles, there is an opportunity to test and develop spatial skills, although the role alone serves little purpose.

As a participant, as an interpreter, we can say that any data visualization solution develops the spatial perception intelligence factor, in its broader sense: perception and processing of what is seen in the context of the appropriate technique. Even in narrow sense, the recognition of perpendicular elements is possible in some specific cases, such as in a 3D pie chart, where one of the data represents just 25 %; or in 3D diagrams, where the axes of the data are represented. The visualization component can also be improved with the simplest line chart if the task is to extrapolate to future values. A waterfall diagram can also be used to make such estimates. The third component that can be developed in this role is spatial relations, which can be developed for surface and 3D point cloud diagrams when trying to imagine suitable views of the represented dataset. However, imagining a suitable intersection is again a task requiring spatial visualization.

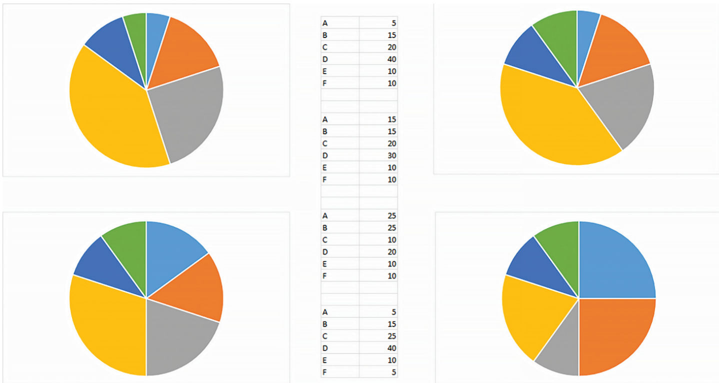
The creative role, if done with the help of a computer, may well not develop any spatial skills on its own. The data selected can be automatically translated into a visual language if you know how to use the software. But here it must be necessary to decide whether the visualization created is suitable for the desired purpose. In other words: we can create data visualizations with our programs using a minimum of spatial capabilities. However, the goal should be to enhance spatial perception, and we believe that this can be achieved through specific development tasks, examples of which are given here. Unless otherwise stated, the numbers on the diagrams are suppressed in the exercises, the exact values are not shown, only estimates can be made.

Source: own edit

Figure 1. Visualization exercise: switching between column and pie charts.

Tasks to improve spatial visualization may include items that require to switch between different representational styles, and for this purpose standard representational methods may be equally suitable. For example, given cumulative bar and pie charts, and the representatives of the same data should be selected (Figure 1). Later, the task can be extended to switch from column → cumulative column → pie chart. To save space, only 2-2 alternatives are shown in some examples presented, but it is recommended to give at least 4-4 alternatives and select those that match.

To switch between the numerical and visualization modalities mentioned earlier, i.e. to visualize numerical data, the following task can be used: given sequences of numbers and some pie charts (Figure 2). Pair the data with the corresponding diagram.



Source: own edit

Figure 2. Visualization exercise: pairing data sequences and pie charts.

Similar problem can be constructed but with bar charts (Figure 3).

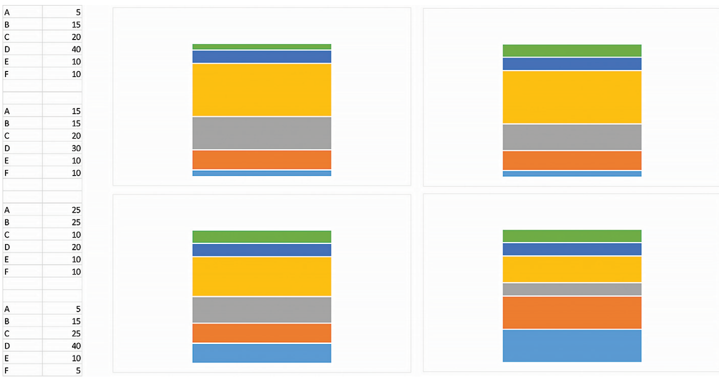


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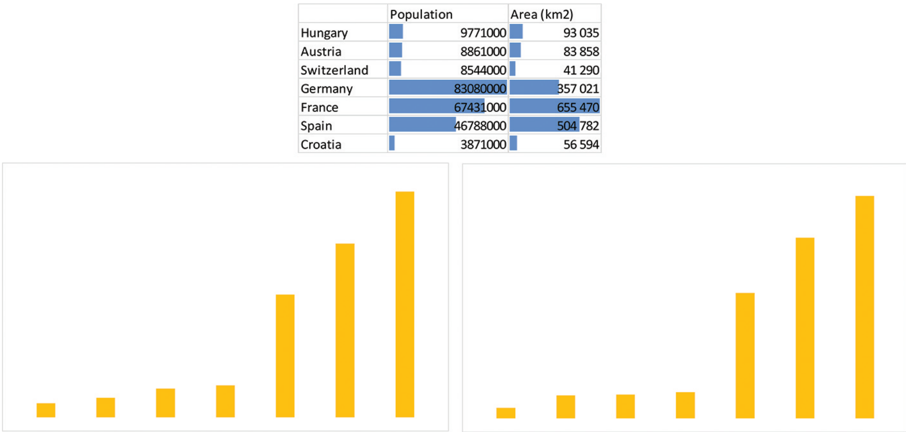
Figure 3. Visualization exercise: pairing data sequences and bar charts.

Some exercises using non-standard data visualization methods, which also improve the visualization factor, are also presented, as these methods are easily available through software. On the other hand, students can easily come across this kind of visualizations in the context of other subjects or when searching in statistical databases, i.e. they are not necessarily strange to them.

For example, a visualization problem with bar charts may be reformulated into data bars. In this task, bar charts and data bar visualizations describing the same data need to be matched (Figure 4).



	Population	Area (km2)
Hungary	9771000	93 035
Austria	8861000	83 858
Switzerland	8544000	41 290
Germany	83080000	357 021
France	67431000	655 470
Spain	46788000	504 782
Croatia	3871000	56 594

Source: own edit

Figure 4. Visualization exercise: switching between bar chart and data bars.

Converting line diagrams into radar charts is often not meaningless, so we have another opportunity to improve visualization. This type of visualization can be solved with a similar visualization strategy as the cumulative bar chart ⇔ pie chart conversion (Figure 5).

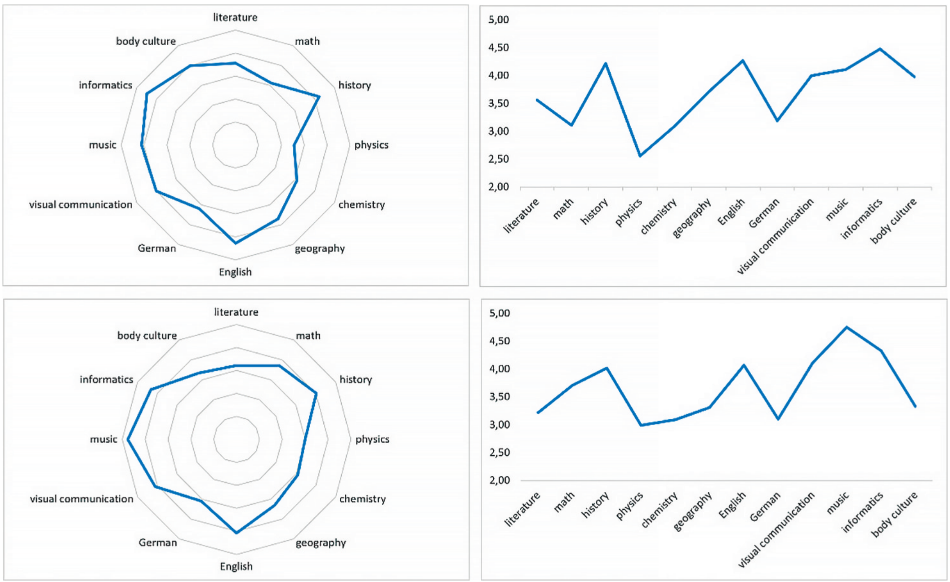


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Figure 5. Visualization exercise: switching between line chart and radar chart.

The following are recommendations for exercises to improve mental rotation.

First, we cite one of the tasks from (Hall et al., 2021): given a set of pie charts, select those that represent the same data structure. Here, the diagrams do not contain numbers either, but they can be rotated relative to each other. The effect of a 180-degree rotation about a line in space is the same as if the data series is plotted in reverse order. Again, the goal in this exercise may be to find the pairs of the same representations. The task is very similar to the one of selecting identical flags from a set of given flags (Figure 6).

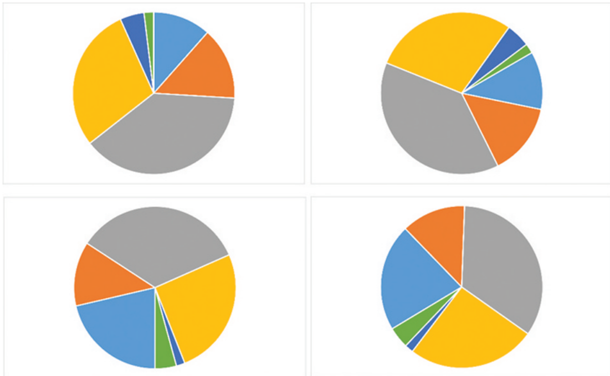

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Figure 6. Mental rotation exercise: find the pie charts representing the same data.

You can make the task more difficult in several ways: you can change the order in the data set to produce a pie chart that is a rearranged version of the original one. You can also combine the task with changing the display style. At these levels, mental rotation is used along with the visualization ability (Figure 7).

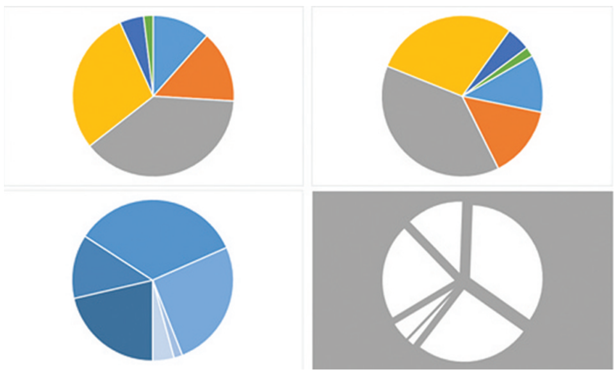

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Figure 7. Mental rotation exercise: find the identical pie charts.

A simpler exercise to practice mental rotation is to replace bar charts with band charts (Figure 8). Actually, this is a transposition but can also be done by a spatial motion.

Source: own edit

Figure 8. Mental rotation exercise: switch between bar and band charts.

Through data visualization, we can develop creative thinking. This is also manifested in the need to find new solutions for non-standard data sets that cannot be adequately visualized using conventional (by which we mean previously learned) tools. Such non-standard representations include Sankey diagrams for complex process representation, or the beanplot technique for visualizing large amounts of clustered data, or the boxplot technique for visualizing linked data series and value boundaries (Kampstra, 2008). It is therefore also developmental to recognize the limits of each technique; to look for new methods and to compare our ideas with their implementation to decide whether they are applicable. Furthermore, deciding whether a visualization is appropriate (i.e. which is the easiest to interpret, most informative, etc.) is a complex task, which students also need to be educated to perform. It is also important to note that we do not have to accept the visualizations offered by the programs, but can change them in many ways, as settings as style, and thus make our data visualizations more informative and/or aesthetic.

5. Summary

In this paper, we have given a theoretical summary of the concepts of data visualization and spatial intelligence and provided a program for improving spatial vision through data visualization. We propose that the development of spatial intelligence, which seems to be neglected, can be given a new impetus by using statistics, more specifically its data visualization content. We believe that improved spatial intelligence can help students both to succeed in their studies in higher education, to reduce drop-out rates, and equip them with a useful skill for their whole life.

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A statisztika és térlátás összekapcsolása

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Absztrakt. Korábbi kutatásaink bebizonyították, hogy a magyar oktatási rendszerben az elmúlt évtizedekben a térgeometria oktatása jelentősen visszaszorult. Ez a folyamat nemzetközi viszonylatban is jól ismert. A matematika oktatásában tehát a téri intelligenciát fejlesztő egyik legfontosabb elem redukálódott. Mindez azért is sajnálatos, mert a téri képességek fejlesztése bizonyított módon pozitív hatással van a természettudományos és STEM tárgyak elvégzésére, így a lemorzsolódás áhított csökkentése irányában hatna. Ezen túl számos tanulmány érvel amellett, hogy a téri képességeknek nagyobb szerepet kéne kapnia a földrajz, térinformatika, szociológia, orvostudomány, üzleti tudományok és marketing oktatása során is. Emellett kutatások igazolják, hogy a téri intelligencia javításával például sikerek érhetők el még a diszlexia ill. a diszkalkulia terápiájában is.

Mindezekkel szemben a statisztika iskolai oktatásával kapcsolatban a fentiekkel ellentétes tendencia, annak fokozott térnyerése figyelhető meg. Ezért a jelen előadásban azt vizsgáljuk, hogy milyen módon lehetne a statisztikai feladatokat egyúttal a térszemlélet fejlesztésére felhasználni. Javaslatokat fogalmazunk meg azzal kapcsolatban, hogy milyen feladattípusok és milyen módon lennének alkalmasak a téri intelligencia faktorainak fejlesztésére. Ezen belül külön foglalkozunk két képességelemnek, a mentális forgatásnak és a vizualizációnak a lehetséges fejlesztésével.

Kulcsszavak: statisztika oktatása, téri intelligencia, vizualizáció, mentális forgatás, lemorzsolódás csökkentése

Learner-Generated Drawings in Mathematics: Who? When? How?

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Abstract. Learning as a generative activity involves making sense of the information to be learned through mental reorganization and integration with prior knowledge, thereby enabling individuals to apply acquired knowledge in new situations. It is promoted in recent times through different learning strategies. One of the strategies is a drawing strategy which involves the construction skill which should also developed in math classes.

This paper presents the results of the initial knowledge assessment conducted at the Faculty of Civil Engineering of the University of Zagreb, in October 2022 with the aim of gaining insight into the development of construction skills as a mathematical activity. In this paper, the term "construction" refers to a mathematical activity that is performed in the mind with ideal objects and includes the activity of finding the sequence of construction steps while respecting the properties of various geometric concepts. The realization of the mathematical idea itself is carried out through drawing, which in this research is carried out with the permitted aids of a compass and a ruler (although in general, in mathematics lessons, drawing can be carried out with the help of other aids).

The paper analyzes the results of the constructions of 111 first-year undergraduate students. Mathematical development of construction skills is of particular importance for students of technical fields because spatial problem situations from the profession are solved by applying constructive procedures through deepening and expanding existing geometric knowledge and concepts about space from the lower and secondary school. Along with the discussion of the test results that indicate that there are significant deficiencies in the adoption of fundamental concepts from plane geometry at the secondary level of education in Croatia, we make proposals for the selection and design of construction tasks in the field of 3D geometry that promote student learning and mathematical modelling through the use of precise mathematical explanations for the constructive procedures that are carried out. It is important to emphasize that the ability to perform geometric constructions accurately and independently, even in the age of increasing digitization, is particularly important for the development of spatial reasoning and should not be underestimated.

Keywords: geometric reasoning, drawing, representation, mathematic modelling, visualization

1. Introduction

The rapid development of modern society under the influence of technology indicates the importance of expanding the learning and teaching mathematics through the simultaneous development of various knowledge, skills, and abilities (Blum, 2015; Herbst et al. 2017; MZO, 2019; Weigand et al., 2018). One of the needed skills that support learning, not only in mathematics but in other STEM disciplines as well, seems to be in connection to drawing activities (Fiorella & Meyer, 2015; Van Meter & Garner, 2015). On the other hand, since the drawing construction processes imitate the modelling practices important in STEM disciplines, it has been shown that the learner-generated drawings support a constructive process of thinking in action important (Kuzle, 2019; Rellensmann, et al., 2017; Wu & Rau, 2019).

Therefore, the focus of this paper is on the stronger promotion of the use of drawing and constructing processes in mathematics, at all educational levels, particularly in connection to the development of spatial reasoning in mathematics (Kovačević, 2017). Also, drawing activities can be used in mathematical modelling due to the fact that they can simplify challenging transfer processes between reality and mathematics that are at the core of solving real-world problems (Blum, 2015; Niss, 2012; Rellensmann et al., 2017). However, constructing activities are particularly important when it comes to learning specific geometric concepts because drawing activities have been designed to lead to a deeper understanding of the concept through its various representations and can therefore be used to teach pupils to view the world around them more mathematically (Kovačević, 2017). Unlike the technical fields where constructing process means making plans and designs for building physical objects and machines made by engineers and architects, in geometry the term “construction” refers to a mathematical activity that is performed in the mind with ideal objects and includes the activity of finding the sequence of construction steps while respecting the properties of various geometric concepts (Filler, 2019; Weigand et al., 2018). Depending on the used tools or instruments, the realization of the mathematical idea itself is carried out through various types of representations. The most common representations in geometry were graphical (besides traditional tools for making drawings in mathematics like compass and ruler, various educational dynamic geometric software are also available today), although today constructions in mathematics can be made by other tools as well (e.g. Cuisenaire rods manipulatives and straps, rubber bands and geoboards…)

In general, in Section 2, I will briefly consider only three basic interrelated questions: **Who? When**? and **How**? in connection to the use of learner-generated drawings in high school mathematics. Namely, geometry seems to be a suitable tool for the use in mathematical modelling for it can naturally facilitate the transfer processes between reality and mathematics and provide an introduction to logic and the process of proofs. Therefore, all over the world the entire high school geometry today balances between its strong need to be technically oriented and applicable when it comes to solving real-life 3D problems, and at the same time the traditional development of school geometry only as a tool for developing logical mathematical thinking through the application of 2D Euclidean geometry.

Starting with the more or less classical geometry content: proving, constructing, problem solving, concept forming (planar and spatial figure, area and volume, trigonometry functions, congruence and similarity transformations), today in different national high math school programs, one can only notice more or less addition or subtraction of the mentioned content through a greater emphasis on one or another subject area. Unfortunately, in recent years it seems that in Croatia construction activities in connection to geometry are supported only at lower educational levels, while modelling real-life 3D problems at high school mathematics are strongly focused only on the development of students' algebraic skills, while at the same time neglecting the importance of simultaneous development of geometrical and spatial reasoning (Gusić, 2011; Kovačević, 2017; Terzić, 2019).

Hence, getting out of the traditional geometry is needed in 21^{st} but the basic question seems to be how to do it. How do visual aspects of computer technology change the dynamics of the learning mathematics i.e., what are the didactic changes in modern school geometry that result from the use of new tools in teaching based on the new social demands for stronger skill development and the tighter connection of geometry with other areas within and outside mathematics? How and when to apply the new multimedia learning trends that steam from the combination of learning from words and pictures into mathematics? What aspects of the use of different types of imagery and visualizations are effective in mathematical problem solving at various levels? How can teachers help to make connections between artistic visual imagery and conventional mathematical processes and notations? Should constructions in mathematics still be done with a compass and ruler? Should a change in the application of tools change the way constructions are associated with proof activities in mathematics? Who should give instructions on the use of different tools in drawing activities in mathematics classes, mathematicians or computer scientists?... Although the answers to some of the above questions are already well known, many questions still need to be detailly analysed in detail before any simple definitive answer can be given.

After a short discussion, in Section 3, the results of the initial knowledge assessment conducted at the Faculty of Civil Engineering of the University of Zagreb, in October 2022 are given. The goal of the assessment was to determine students' prior knowledge about basic geometric concepts and the ability to constructively solve mathematical problems which seems to be of importance in STEM fields. Drawing strategies are in focus when teaching descriptive geometry aiming to strengthen 3D geometry content-related and process-related competencies.

2. Drawings as a part of construction process in mathematics

It is well known that teaching (as well as learning) mathematic is a demanding process, mostly due to its variety in content (algebra, geometry, probability, statistic...). Today different strategies are used aiming to overcome the difficulties that students encounter when learning mathematic (Blum, 2015; Niss, 2012). Among them special emphasis is placed on strategies that facilitate the integration of different forms of representations in the domain of mathematics (textual, visual, algebraic, symbolic. . .). The importance of multiple representation usage, as well as its implementation into teaching process is strongly supported all over the world mostly pointing how ". . . representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships. . . New form of representation associated with electronic technology create a need for even greater instructional attention to representation' (NCTM, 2000 p. 67).

One of such promising multimedia learning strategy is *a drawing strategy* (Fiorella & Mayer, 2015), a type of strategy traditionally used in the domain of geometry, for constructing is one of the fundamental activities in geometry (Weigand et al., 2018). The choice to use the term *learner-generated drawing* in the title of this paper, instead of simpler term *drawing*, is due to many math educators who use this term mostly in connection to currently important topic in mathematics education, mathematical modelling. For example, Rellensmann et al. (2017) point out that *learner-generated drawing describes the process and the product of generating an illustration that corresponds to the objects and relations described in the task*, while Van Meter and Garner (2005) define learner-generated drawing in general *as a strategy in which learners construct drawing(s) to achieve a learning goal*. Such drawing description actually describe construction process that was traditionally in geometry made by simple tools like compass and ruler. However, due to overall social and technology progress it is now strongly supported by variety of tools including various dynamic geometric software programs.

On the other hand, the fact that drawing activity is strongly supported by the use of graphic representation and visualization has often caused a lot of problems when it comes to learning mathematics, especially for students with lower mathematics ability (Cooper et al., 2018; Krawitz & Schukajlow, 2020). Modern implementation of spatial geometry problems into lower educational levels has only intensified the problem of using 3D images in mathematics for even simple 3D construction problem may be a challenging task for students (e.g. various types of projections (parallel or central) can cause simple planar objects like squares to be presented as parallelograms, or circles to be presented not only as ellipses but as parabolas or hyperbolas as well). This is probably one of the reasons why in recent years many mathematics teachers not only avoid the use of drawing activities in teaching math at higher educational levels, but also avoid the use of drawings themselves. The mathematical theory behind projections opens a whole new field to be learned in school mathematics that is often not adequately supported in teaching mathematics due to the large reduction in geometry curricula worldwide over the decades (Kuzle & Glasnović Gracin, 2020). On the other hand, there is a growing number of mathematics teachers who simply use ready-made graphics (illustrations/videos) as support for emphasizing the importance of a specific mathematical concept in the social and professional life of students. They often aim to replace the trivial (problematic) 3D graphic representation as soon as possible with abstract, mathematically more operable representations, such as symbolic or algebraic. However, the fact that the wealth of different representations of mathematical concepts allows mathematics to be present in all sciences should not be ignored, and all types of representations should be continuously connected and supported in the practice of teaching and learning. For visual ability is not only a strategy, or a type of thinking but also a supporting tool for obtaining logical chain of reasoning to develop the formal analytical abilities crucial in mathematics. The way in which abstract mathematical ideas are presented is crucial to how an individual will understand them, and more importantly, use them to solve problems in everyday and social life through the application of modelling (NCTM, 2000).

Drawings have often been used in mathematics lessons without words, or any instructions on how to read them “mathematically correct”, or how to analyse them or maybe construct them by yourself, for drawings have been typically treated like an adjunct aid whose effectiveness is related to the provision of external support (Van Meter & Garner, 2015; Presmeg, 2014). Consequently, the accompanying “beneficial effect” of drawings in promoting students’ learning and problem-solving skills in math is often not automatically appreciated and recognised by all students.

2.1. Who? When? How?

Starting from the already mentioned three interrelated wh-questions the easiest part seems to be in connection to when. Although drawing activities in the past decade addressed different goals from different perspectives (Wu & Rau, 2019; Kuzle, 2019; Rellensmann et al., 2020), modern didactics put strong focus on spiral teaching principle based on a Bruner hypothesis that the basic ideas of the subject can be conveyed to any child, regardless of age or social origin, in an appropriately simple form, based on the way of thinking he/she carries with him/her and the way of presentation he/she understands (Filler, 2019). Today there is a number of studies mostly from educational psychology pointing out to benefits of drawings in various STEM fields (Fiorella & Meyer, 2015; Fiorella & Kuhlmann, 2020; Presmeg, 2014; Van Meter & Garner, 2015; Wu & Rau, 2019) and they should not be neglected by mathematical educators. However, mathematicians should have the last word on teaching mathematics. Or, in case of geometry content, geometry educators.

Firstly, drawing supports the organization of the given task information. Also, students have to reduce the amount of given information by focusing on the relevant pieces of information that are to be represented in the drawing. Rellensmann et al. (2017) also point out that various benefits may be attributed to learner-generated drawing in connection to math problem solving. In general, they conclude that drawing can help students “better ‘see’ mathematical concepts and ideas”, but, in accordance to previous results, the use of drawing strategies does not foster understanding and problem solving automatically. They concluded that, besides the lack of mathematical and drawing knowledge, reduced performance outcome in some tasks may be due to the fact that sometimes learner-generated drawing can also increase learners’ cognitive load. Furthermore, a numerous studies have shown that drawing activities are effective only if they are implemented with instructional support that targets specific goals of the drawing task. Krawitz and Schukajlow (2020) suggested that the improper use of visual-register in mathematics shown by some students might be due to the lack of the mathematical knowledge necessary to proceed and they concluded, as many other educators, that the way the drawing is used determines whether it is useful or damaging, and that more efforts are essential to enable learners to apply drawings appropriately. Also, in accordance to mentioned spiral principle, all educators emphasizes that continuity is the key to applicability of new knowledge.

Let us continue with some further comments on the quite problematic how part. How to use drawings in mathematics? Using tools or simply drawing sketches? What tools are suitable for certain age? ... It is well known that traditional primary focus on teaching is slowly giving more space to the learning process (Blum, 2015) requiring from teachers to find balance between their guidance and students independence learning applying, among other things, different didactic principles. In mathematical literature there is a number of didactical principles whose applicability must be continuously assessed in relation to the goals, knowledge and skills of the student for principles are not the type of knowledge that, once discovered, become irrevocably valid, but must be discussed again and again in a constantly changing world (Herbst et al., 2017). For the time I will focus only on three such didactical principles that seems to be of relevance to current geometry learning processes (Weigand et al., 2018). Hence, the answers to the mentioned three wh-questions, Who? When? How? are viewed in the light of three didactical principles: *Operative principle*, *Spiral principle*, *EIS principle*.

Modern didactics is strongly influenced by different learning theories, often stemming from different psychological practices. Particularly strong is the influence of the constructivist learning theory that "students learn not by passively listening, observing, or otherwise receiving information, but through a process of actively constructing their own knowledge, adapting their cognition to assimilate their experienced world" (Herbst et al., 2017, p. 43). Two inevitable names are two psychologists Jerome Bruner (1915 – 2016) and Jean Piaget (1896 -1980) whose work had some shortcomings in relation to mathematics but was eventually implemented and modified into what appear to be applicable geometric didactic principles.

Operative principle: Piaget research was in the field of children's cognitive development where he studied how a child's mind evolves through a series of set stages to adulthood. He concluded that those stages form an order-defined process that is not necessarily age-oriented. To Piaget, the central concepts of the cognitive theory are internalized (mental) actions called *operations* used to describe thoughts. According to him, all operations are reversible, connectable and composable, thus the same goal can be obtained differently. His work on operations was further developed by one of his pupil *Hans Aebli* (1923 – 1990) who emphasized the importance of educational conditions and teaching in the mentioned process of internalization. Aebli describes the structure of operations by characteristic steps that rely on general psychology of learning and mental processes and are applicable in secondary as well as in primary school. Basically, he points out that the perception is the beginning of any action and that the relation between thinking and acting is theoretically well established. Hence, the structure of operations can be summarized through characteristic steps: *to do, to understand, to internalize* (i.e. make part of one's nature by learning), *to automate* (for further details see Bünning, 2007). When it comes to learning and teaching geometry, some educators further emphasize three main levels in the process of internalization of an operation (Filler, 2019): starting from the *concrete level* and working with concrete objects and materials, the *figurative level* operates with visually represented objects (drawings) and on the *symbolic level*, objects and operations are represented by signs.

More formal description of this principle is given in (Filler, 2019) stating that operative principle *guides a lesson that awakens thinking within the framework of action, builds it up as a system of operatives, and finally puts it back in the service of practical action*, or as. Also, in connection to geometry, Lambert (2012) says that the characteristic of the operational method is the study of objects together with the construction that creates them and the operations that can be applied to them, i.e. the examination of which properties and relations are "imprinted" in the objects by the construction and how the properties and relations behave when the operation is applied.

Spiral principle: The research of an American psychologist Bruner has proven to have decisive meaning for didactics, especially didactics of mathematics and it has been discussed a lot. Spiral hypothesis serves as a starting request implemented in many subject curricula worldwide. Let us state some formal description of this principle implemented in mathematics: *In order for learners to recognize the connecting lines of common threads and networks of relationship in mathematics, the content of mathematics lessons cannot be divided into incoherent areas but must be focused on some fundamental content-oriented ideas that serve as a basis for the development of general mathematical competencies* (Weigand et al., 2018).

Hence, teaching should primarily be directed towards the basic ideas or structures of the subject in question which in the Croatian mathematics curriculum are based on the connections between *mathematical processes* and *domains*, two sets whose interaction is reflected in the stated learning outcome and thus contributes to the acquisition of mathematical competencies (MZO, 2019, p. 7). However, the domains are generally described, so there remain decisive questions in every national curriculum, with regard to specific school reality, which are actually the fundamental content-oriented ideas, how they are implemented within the domains and with which basic didactic principles they are aligned in everyday practice. It seems that its application in connection to geometric content is somehow problematic for many decades in many countries, for some studies show that geometry teaching has often been reduced to a somewhat eclectic mix of activities (Kuzle & Glasnović Gracin, 2020).

In order to more closely implement the spiral principle of the curriculum, educators further derive some guiding principles when it comes to teaching mathematics and geometry as well (Filler, 2019). Hence, we have here two more sub-principle to be followed in teaching and learning mathematics:

- *Principle of anticipatory learning (ger. Prinzip des vorwegnehmenden Lernens)*: The treatment of a field of knowledge should be introduced at earlier levels in a simple form.
- *Principle of resumability (ger. Prinzip der Fortsetzbarkeit)*: The selection and treatment of a topic at a specific point in the curriculum should not be ad hoc

but should be done in such a way that it can be expanded to a higher level. Superficial didactic solutions that later require rethinking should be avoided.

EIS principle (Enactive – Iconic – Symbolic): In addition to the contribution to didactics through the spiral curriculum, Bruner's work is also important in relation to the exploration of the three modes of representation in the acquisition of knowledge and ability. These three representations type bear some similarity to the already mentioned internalization levels of operations (Filler, 2019) actions, pictures and symbols, or:

- Enactive representation (acquiring knowledge through action)
- Iconic representation (acquiring knowledge through drawings)
- Symbolic representation (knowledge acquired using conversational and mathematical language, i. e. language-based "knowledge").

EIS principle emphasized that in the teaching process it is important to strengthen the possibility of constant change between all three ways of representation, always in both directions, in order to support the development of the cognitive concept. Students need to be able to move freely within the enactive and iconic representation in order to achieve the symbolic understanding. Often need for action allow students to learn what they cannot initially say or draw, for example rotation without turning, congruence without laying one object over another may cause obstacles to pupil's learning. There is a huge similarity between EIS three levels of intelligence with Aebli's three levels of internationalization, but EIS levels are mutually related to each other, while Aebli's levels are three successive levels (like the van Hiele levels of geometric understanding). The same actions can be assigned to different levels at different ages and levels of knowledge. For example, making of drawings can be assigned to the iconic level as the first abstraction, but as a concrete action it can also be located on the enactive level. Similarly, the use of the computer – e.g. depending on familiarity with it – have an enactive, iconic or sometimes symbolic character, depending on an individual.

2.2. Using tools in solving problems

Dynamic software for geometry is proving to be more successful (and faster) in solving a range of problems and it's sometimes taking a role that does not belong to it, for it is nothing more than a tool in a process, offering a product, in this specific case iconic, graphic, representation. As some researchers have shown, it seems that drawing was (and still is) incorrectly detected as an adjunct aid in mathematics, for drawings in mathematics should be about process and product, in short constructing (though some may prefer the term modelling) (Van Meter & Garner, 2015).

But the use of aid requires from learners to have well-developed geometric (not only spatial) reasoning with a wide range of complex geometric concepts. Meaning, learners should be able to see the world around them from a geometric perspective, they should recognize geometric concepts and bodies in their environment and be able to analyse it with regard to their function and meaning (Weigand et al, 2018).

A few words on the use of sketches seems to be needed as well. A sketch is usually a freehand drawing with no requirement for scale but presents some essential information about the displayed image (e.g. consists of positional relationships, label names, visibility that should reflect the corresponding view. . .). Unlike drawings obtained by constructing, drawings obtained by sketching are more graphic oriented towards the creation of "image" for sketches neglects the construction process involved. However, sketches can be used as a starting point for the construction step by step process. Unfortunately, construction processes have been increasingly neglected in recent years, mostly due to lack of time. In this way teaching process does not adequately support students' understanding of mathematical concepts and relationship through its various representations. Hence, not teaching the use of certain type of representation, or even taking some representations for granted in the teaching process, may cause serious problems in this continuous learning process (Cooper et al. 2018; Krawitz & Schukajlow, 2020; Leopold & Mayer, 2015). From a didactic point of view, the proposal is to use sketches only after the concept has been properly developed at some educational level (spiral curriculum supports earlier introduction of the concept), therefore the didactic use of sketches varies from primary to secondary school. Sketching should also be promoted in math curriculum through simple teachers' instruction (sketching parallel or perpendicular lines, or sketching basic solids supporting (only in higher grades) the so-called parallel projection (or oblique projection) (e.g. eight edges of a cube in such graphic representation should be presented by two groups of four (only) visually parallel sides, and length in each group should be visually (at will) equally abbreviated).

Construction as mathematical activity: In construction tasks, the goal is to create the target configuration, starting with the given initial configuration, creating the individual design steps that vary in connection to the approved tools. Both configurations are ideal (mathematical objects) and they consist of a set of geometric objects (e.g. point, circle, line,. . .) and a system of conditions (e.g. the point lies on the line,. . .). In addition to finding a solution (the actual realization of constructions is always only a representation of the ideal construction), it is also important to present it. In doing so, the solution is additionally mathematical analyzed: whether there is a unique target configuration or there are more possibilities, or perhaps there is no solution (in relation to the given starting configuration). In contrast to traditional approach, modern geometric didactics emphasize that the whole description of the construction (step by step) is also an important part of solution for it supports cross-curricula algorithm approach (see e.g. Weigand et al., 2018). Namely, it is of didactic importance to write down every step of the construction process (similar to the steps involved in solving an equation) because unlike algebraic tasks, the constructive task is performed with numerous "overlaps", and in the end it is not possible to see how the finished structure was created (Weigand et al., 2018). The construction tasks generally contain three fundamental phases which are commonly recognized also in problem solving, hence their use in the classroom is a way of promoting mathematical modelling (Niss, 2012; Blum, 2015; Rellensmann et al., 2017):

a) *heuristic phase*: understanding the task, developing a solution plan, . . .
b) *algorithmic phase*: carrying out the construction, design description (depends on the used aid)
c) *analytical phase*: justification of the correctness of the construction, considerations of the solvability and uniqueness of the solution.

Using tools: Hence, in construction process the realization of the mathematical idea itself is carried out through graphical representation – drawings, either a picture on a sheet of paper or on a computer screen (i.e. image), depending on the aids allowed in the task. Compass and ruler have been the tools on which the elementary geometry of the plane is based since Euclid and his book “The Elements” (ca. 300 BC), though today they are supplemented and expanded by other aids and tools such as Geodreieck[1] or geometry software (didactical or professional, 2D or 3D). The construction process, (i.e. its steps) always depends on the tool, hence requires from users different concept networking (this term will be discussed in more detail in the next section). Today, the use of computers in the constructing process is unquestionable, but some teachers have doubts about today’s use of compasses and rulers in geometric constructions. However, compass and rulers support the concept introduction in the geometry. For example, transferring equal length was significantly facilitated by the compass for unlike rulers, compasses “transferred equal length in all directions”. This naturally creates a strong connection (both mental and physical) between “circle (as a mathematical concept)” and “compass (as a mathematical tool)”. Regarding teaching geometric constructions with both compass and rules and computer software, I would discuss the advantages of rulers and compass only in connection to learning concepts and simple construction for gaining insight into geometrical reasoning. Both computer and compass are geometric aid. Their use requires skill that should also be taught in school. However, using compasses correctly is an easier to learn than using a computer software. After specific concept knowledge is gained, skills are easier to develop. Namely, as Weigand et al. (2018) point out, in geometry, compass-and-ruler constructions are not based on a practical but on a theoretical interest. The goal in didactical sense is on theoretically exact construction, for the goal is on creating mental objects through mental operations using some idealized operations (drawing circles, lines. . .). A construction is “exact” in the sense of exact only in the theoretical sense when dealing with ideal objects under ideal operations.

As already mentioned, construction tasks can have different functions in the classroom. Apart from problem solving tasks, constructions in geometry are also used in connection to learning geometric concept. Namely, contemporary geometric didactic emphasizes that learning geometric concepts is a process designed to lead to the understanding of the concept. It includes i) *building appropriate images* about the concept, ii) *acquiring knowledge* and iii) *acquiring skills* related to the concept. These three activities mentioned also in the Table 1 are important

[1] Geodreieck (Geometrie Dreieck, eng. gemetric triangle) is a specific type of modern geometric aid made of clear plastic; an integrated combination of ruler and protractor in the shape of a right-angled isosceles triangle. It is used for measuring and drawing angles and facilitating the drawing of orthogonal lines.

aspects of learning geometric concepts and they constantly occur in an ongoing interrelationship (Weigand et al., 2018, p. 91). However, the included activities are often treated separately in learning outcomes, and some are even only related to specific educational level (though they need constant interaction to support each other):

Table 1. Learning geometric concept.

Building an appropriate image[2]	**Acquisition of knowledge**	**Acquisition of skills**
• to act • to perceive • to verbalize	• properties • relationships between properties • relation to other concepts	• to construct • to calculate • to solve problem

In other words, understanding a geometric concept also includes skills in dealing with the concept. When it comes to geometric skills, the first thing that comes to mind is construction, but the ability to calculate (distance, area, volume. . .), and problem-solving are included as well.

Considering the previous discussion, we can now give the answers to the previous questions, though they have already been implemented in the currently valid spiral math curriculum. However, the focus on constant networking of content and ideas on a cross-curricular basis should not diminish the importance of understanding the diversity within the subject itself: diversity in teaching and learning as well. This naturally make it harder for the teachers, for they are the guidance in the learning process.

Today's tendency is on building application-based curricula but in order to be able to apply some knowledge one has first to gain it, to understand it on the basis level. Only if the foundations of knowledge are solid is it possible to upgrade them. The teaching and learning of geometry (or any other branch of mathematics) should be based on the understanding of its basic concepts, structures that represent knowledge about them as well as skills in working with them through the application of their properties. The understanding of these concepts is then shown in the lesson by the fact that students can operate with these terms in the context of problems. One guidance for understanding geometric concept complexity was offered in Weigand et al. (2018) where it is pointed out that part of the understanding of concept is that learners:

- develop ideas about characteristics of properties of a concept and their relationships to each other (build up ideas about *content of the concept*, germ. Begriffsinhalt),
- get an overview of all objects that are summarized under one term (develop ides of *range of the concept*, germ. Begriffsumfang), and

[2]The German term Vorstellung is sometimes used in different contexts in English language hence in this paper I will use the more common term "image" instead of "imagination" due to the context of the paper.

- are able to show relationship between the concept and other concepts (form ideas about the *concept network*, germ. Begriffsnetz).

Following the constructivist approach, the to-be-learned "information" has to be mentally reorganized and integrated with one's prior knowledge in order to be applicable in new situations. However, in connection with the earlier introduction of some geometric concepts in the teaching and learning of mathematics, one should be particularly careful because, as following some geometric educators (Weigand et al., 2018), in elementary geometry lessons there are four categories of concepts, each one having its own complexity and each one requiring different approaches or strategies in the classroom:

- *objects*: line, segment, angle, circle, cube. . . .
- *relations*: lies on, parallel, orthogonal, congruent, similarity. . .
- *transformations*: translation, rotation, mirroring, . . .
- *measurement*: length, angle measure, area (surface measure), volume (solid measure) . . .

In addition, in real-life situations, all planar concepts gain further networking complexity in connection to their visual representation in spatial situations that also need to be taught.

Furthermore, in connection to current mathematics curriculum in Croatia with respect to geometry, it seems that not all previously mentioned didactic principles have been fully implemented in the current textbooks (I may dare to say curriculum as well). For example, in connection with the spiral principle, the implementation of the second sub-principle (principle of resumabilit – the one that serves as the basis for offering a skeletal basis in the teaching of geometry at different educational levels), is not clearly visible through all educational periods (Kuzle & Glasnović Gracin, 2020). Also, in connection to one of five mathematical processes highlighted in Croatian math curriculum, "display and communication" (MZO, 2019), which is again in close connection to geometry and geometrical thinking, none of the learning outcomes (through primary to secondary education level) deals directly with the concept "projection" (for example oblique projection), but some are dealing with *"orthogonal projection"* (for example learning outcome at the level of achievement good at the second year of secondary school states that learner should ". . . explain the mutual positions of points, lines and planes and determines the orthogonal projection of a geometric object" (p. 108)). In achieving this learning goal learners deal (according to the available textbooks and various obligatory or additional educational web sites used in Croatia) with both drawings made by various types of projection: central projection (photos are often included as well), oblique projection or orthogonal projection. However, the fundamental concept theory in connection to "projection" is not given to them (or it is not stated in the curricula as obligatory) and this naturally leads to the expected difficulties in proper acquisition of many (planar and spatial) geometric concepts which can already be observed in mathematically incorrectly (or sometimes just vaguely used) terminology also in recent graduate thesis.

Some modifications of traditional constructive tasks are given in Appendix A by introducing simple modelling tasks based on the initial constructive problem. However, one should be careful when introducing modelling into constructive tasks for modelling tasks are complex and therefore difficult for both students and teachers. They require translations between reality and mathematics that should vary according to educational levels (following spiral principle, it is preferable to use "upgradable" tasks). When solving real-life (3D) problem using the constructive method (of course at higher educational levels, not in primary school), simplified planar (2D) situations (given by constructive of schematic view) should not be offered to students directly as solutions (or a step in a construction process) without discussion, because students should be aware of the simplification carried out in the task. Students need to be taught how "to correctly spatially read" partly schematic drawings of some real-life situations, how to "see" right-angled triangle 3D representation in order to later apply some 2D theorem (e.g. Pythagoras) to a spatial (!) situation.

An example of complexification could be given in all construction tasks (see e.g. Appendix A: What if the areas shown by the view above in both Example 1 and 2 are not flat (i.e. two significant points included in tasks are not on the same sea level) but slanted? What if the river flow p, or the bus route t, are not straight but curved? Does it (or not) change the construction process? How? Why?...). Task 3 in Appendix A involves direct spatial situation and requires from students to implement various concept in solving it. Besides cube-concept knowledge, they need to use constructive knowledge in connection to the used representation (e.g. parallel projection) as well as used isometry transformation (e.g. rotation of one cube side). Further complexity could be introduced by changing not only positions of points A or B but by changing the starting figure instead of cube (in the form of an octahedron or a hypercube...). Solutions can be offered by modelling nets of figures by paper as well. Constructive tasks can further be used as a connecting bridge to various math areas. For example, one can easily combine different solving methods in the mentioned tasks in other to connect algebra and geometry, simply by offering different questions to guide students in their solving process: How can you introduce a coordinate system (planar or spatial, e.g. how can you place coordinate axis) in the tasks from Appendix A to simplify the algebraic problem solving? Then what are the coordinates of the given points?...). In short, students should be active participants in the creation of tasks (both its simplifications or complexifications).

3. Research methodology and results

In this section the results of the initial knowledge assessment conducted at the Faculty of Civil Engineering of the University of Zagreb, in October 2022 are presented. The study included only 111 first-year freshmen during their first week of classes after an introductory lecture on goals and methods used within Descriptive geometry course, though it was conducted on the larger set of 324 freshmen. The overall results were similar, but here we will cover only one representative part (determined purely by alphabetical order).

Methodology: Though the main goal of an initial knowledge assessment was only to gain an understanding of the student's prior knowledge about the use of drawings in mathematics, the construction process always indirectly includes concept understanding. After almost two years of partial online math lessons due to covid 19 pandemic, the idea was to detect a starting point (how well students were familiar with a simple construction activity) in order to modify (if needed) the given learning plan and to meet the ongoing course objectives as successfully as possible (one of the learning outcomes is to train students to use construction strategies using professional CAD software as an aid) for solving spatial construction problems closely related to their profession. The study also examined the relationship between freshmen's prior knowledge of the use of construction strategy in mathematics in accordance with the learner's gender and the type of secondary school education completed. In this analysis, due to the simplicity of the given tasks on one side and the way the results were obtained (students were given 10 minutes to finish three tasks, given no further instructions but to read carefully the given text within each task), I only differentiate the use of construction skill and basic concept knowledge (only related to pure concept definition).

Three simple planar tasks were chosen, each of which focuses on one planar geometric concept: i) equilateral tringle, ii) tangent line and iii) an intersection point. Despite the later obligatory use of computers, in this task compasses and rules were the offered aid in construction process. Students were instructed to solve the assigned tasks on A4 format paper, on which the initial configurations of each task were graphically highlighted.

TASK 1. Point A and line p are given. Construct an equilateral triangle ABC, one side of which lies on the given line p.

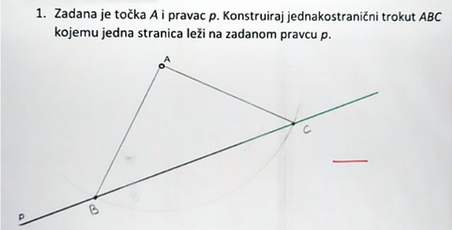
TASK 2. Given a circle and point T. Construct tangents to the circle from point T.

TASK 3. Given are the line p whose equation is $y = x - 2$ and the parabola whose equation is $y = x^2 + 2$. How many intersections are there between that line and the parabola? Calculate the coordinates of the intersection.

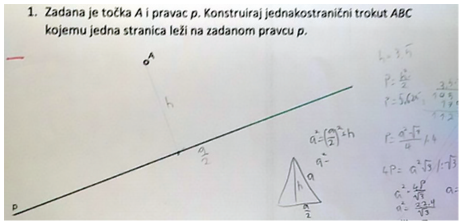
Each task was scored with two points. Scoring was based on the previously introduced didactic approach to learning and teaching the geometric concept. Hence, the first point always referred to the collection of information about the knowledge of the geometric concept (in our case pure concept definition), and the second to the acquired skills in connection to the used concept. Following the idea that teaching of a concept consists of measures that initiate and control the process of learning a concept, we focused on the distinctions in the acquisition of knowledge and skills acquisitions related to a particular concept, assuming that appropriate mental representatives of the analyzed concepts were already developed.

Results: One may note that only the first two tasks were aiming directly at the constructive skill check, while in the last task only calculation skill was required from students. Some examples of students' solutions of Task 1 are given in Table 2.

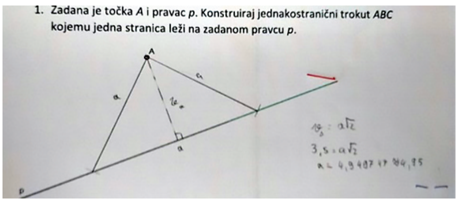
Table 2. Some of the students' result from Task 1.



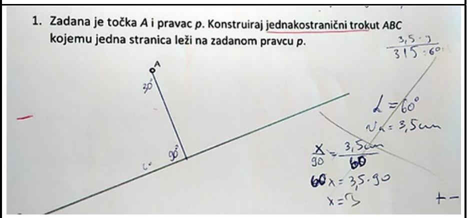
Student 1: scored 0 for knowledge, 0 for skill



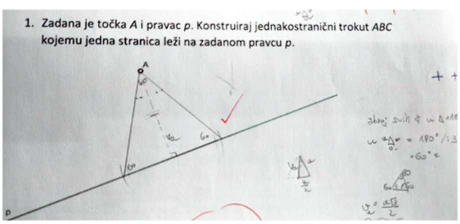
Student 2: scored 1 for knowledge, 0 for skill



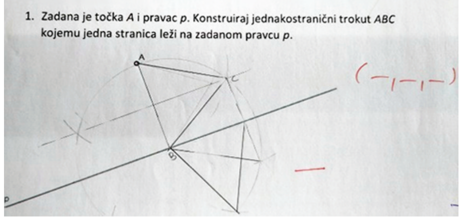
Student 3: scored 1 for knowledge, 0 for skill



Student 4: scored 1 for knowledge, 0 for skill



Student 5: scored 1 for knowledge, 1 for skill



Student 6: scored 0 for knowledge, 0 for skill

As in the classroom the difference between content knowledge and content skill is not covered by the learning outcome (in accordance with Table 1: Learning geometric content), the obtained results are further classified only for the purposes of this paper, in class each task was scored true/false. This further classification aims to highlight not only the problem of understanding the use of a constructive approach in mathematics (regardless of the tools) but also the problem of basic misunderstanding of the concepts (their content, range and networking) introduced at lower educational levels. Although it is debatable whether (surely not in an applicable way), for example Student 3 understands the meaning of the used notation for an equilateral triangle, he was scored with 1 point for knowledge in this study (for the SSS (side-side-side) theorem uniquely defines given triangle, and he/she further understands the given relation "lies on", though he didn't use the equality of angles as well, neither in notation nor in construction). Obtained results are presented in Table 3 and they are unsatisfactory in every way (we expected bad results, but not so bad).

Table 3. Results obtained for Task 1.

			Knowledge 1		**Skill 1**	
	Frequency	(%)	Frequency	(%)	Frequency	(%)
M	68	61 %	14	21 %	4	6 %
F	43	39 %	7	16 %	2	5 %
gymnasium	100	90 %	17	17 %	4	4 %
TVS[3]	11	10 %	3	27 %	2	18 %
TOTAL	**111**		**20**	**18 %**	**6**	**5 %**

As shown in Table 3 only 5 % of students correctly solved this task, which is part of curriculum contents of the 5th grade in Croatia, compared to 18 % of students who accurately presented (only) basic concept knowledge. Hence, this may suggest that just basic definition concept knowledge (in connection to equilateral triangle) is not sufficient to be further applicable, i.e. it does not make concept understandable (or useful) for students. One of the reasons for these poor results may be the shortness of time for solving all the tasks, since many students have not used geometry tools for a long time, so they did not manage the tool well. That may be due to the fact that TVS students' performance was slightly better than the high school (gymnasium) students, both in knowledge (27 % respondents of TVS solved it correctly compared to 17 % respondents of gymnasium students), and skill performance were only 18 % respondents of TVS students correctly solved the first task compared to 4 % respondent of gymnasium students (who as presented in this paper were the majority enrolled at this technical university). Also, as shown in some examples in Table 2, many students were oriented towards the use of calculation skills even though this task specifically ask them to construct the solution.

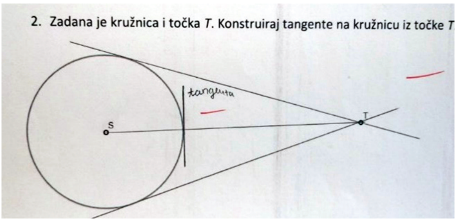
In connection to the second task the results are slightly better (though constructively task was significantly harder) probably due to the fact that only this specific construction was repeated during the introductory lecture three days earlier (Table 4). However, the results indicate the problem in connection to concept relations ("from point T") as well as it is shown in Student 7 and Student 12 results.

The overall results for Task 2 are given in Table 5. In this task (which in this specific case was presented to students few days before the initial assessment took place) gender difference results are interesting in connection to knowledge and skill performance. Namely, the results are in favor of female respondents, for 77 % of female were scored by 1 point in pure concept understanding (tangent from a point) in opposite to only 63 % male respondents. However, in connection to further skill performance, males were significantly better; 24 % of male respondent solved the task completely in compare to only 12 % of female respondent. Here naturally, probably due to the calculation complexity, no respondent used calculation to obtain the result. The overall results were significantly better in Task 2 than in Task

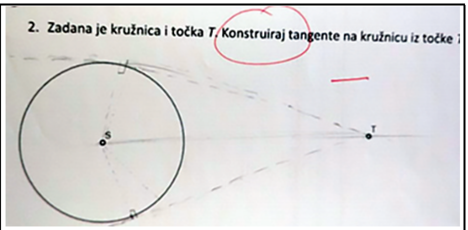
[3]Technical vocational schools

1 on account of knowledge overall (only 19 % correctly solved Task 2, compared to 68 % respondent who showed correct basic concept knowledge. Here again are all results in favor of TVS students (it can be also due to the small number of TVS students), but still only slightly better in overall scores that also included skill usage.

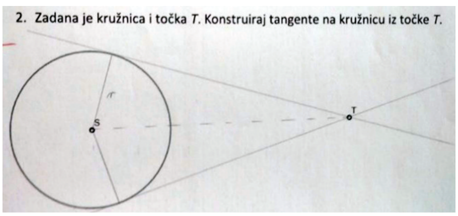
Table 4. Some of the students' result from Task 2.



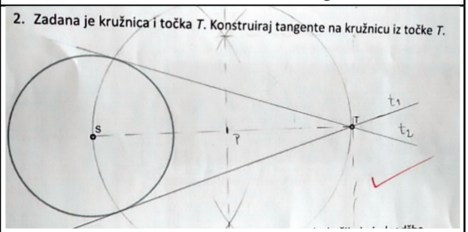
Student 7: scored 0 for knowledge, 0 for skill



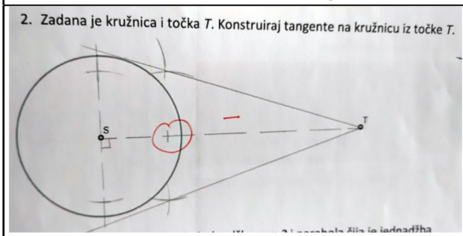
Student 8: scored 1 for knowledge, 0 for skill



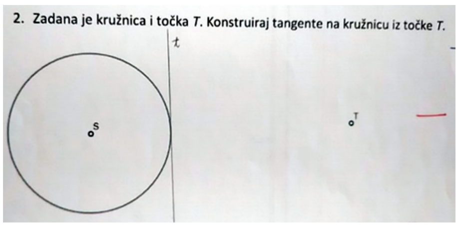
Student 9: scored 1 for knowledge, 0 for skill



Student 10: scored 1 for knowledge, 1 for skill



Student 11: scored 0 for knowledge, 0 for skill



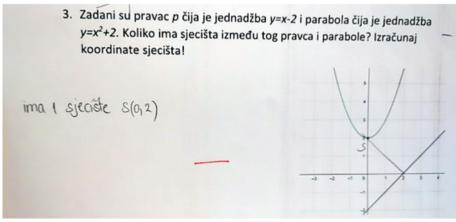
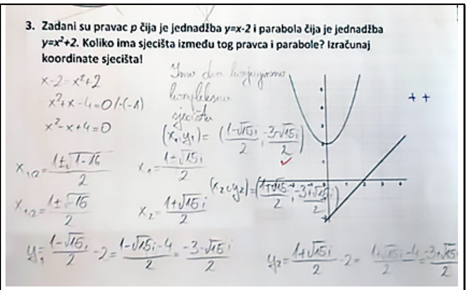
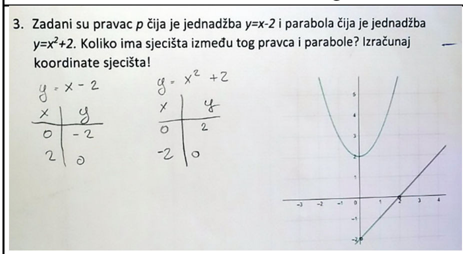
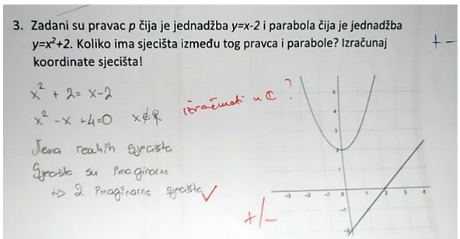
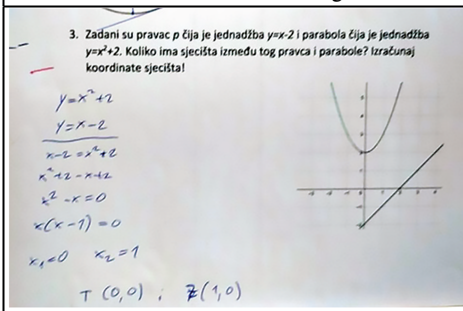
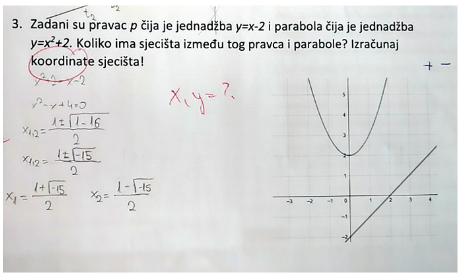
Student 12: scored 0 for knowledge, 0 for skill

Table 5. Results obtained for Task 2.

			Knowledge 1		**Skill 1**	
	Frequency	(%)	Frequency	(%)	Frequency	(%)
male	68	61 %	43	63 %	16	24 %
female	43	39 %	33	77 %	5	12 %
gymnasium	100	90 %	67	67 %	18	18 %
TVS	11	10 %	8	73 %	3	27 %
TOTAL	**111**		**75**	**68 %**	**21**	**19 %**

The last task was primarily concept-oriented aiming to connect students' prior knowledge on solving equations with the geometrical point of view on intersections. It is important for students to perceive that in connection to complex numbers in geometry "invisible is often of the same importance as visible". As for the concept of an intersection point, a unifying approach to real and imaginary points on objects often facilitates the variations in construction process (for example a line intersects a conic in exactly two points which can coincide, be real and distinct, or be imaginary pair points; a plane intersects a circular cone in a conic which can be either ellipse, or hyperbola or parabola depending on its position to cone generating lines). Constructive treatment of graphically different conics is often similar (ellipse and circle being a closed curves, parabola and hyperbola being respectively one- and two-branched open curves) and observing their generalizing properties is one way of mathematization and facilitation of learning process.

Table 6. Some of the students' result from Task 3.

<table>
<tr><td>
</td><td>
</td></tr>
<tr><td>Student 13: scored 0 for knowledge, 0 for skill</td><td>Student 14: scored 1 for knowledge, 1 for skill</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Student 15: scored 0 for knowledge, 0 for skill</td><td>Student 16: scored 1 for knowledge, 0 for skill</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Student 17: scored 0 for knowledge, 0 for skill</td><td>Student 18: scored 0 for knowledge, 1 for skill</td></tr>
</table>

Unlike the previous two tasks, where it was not possible to use the skill without knowledge of the concept, in this task some students showed the development (up to a certain level) of the appropriate skill related to the task, but they did not have developed knowledge of the corresponding concepts related to this skill (Student 18). Interestingly, almost all participants know the procedure (only a few exceptions like Student 13 or Student 15), but many failed in this simple calculation (Table 6). This may suggest that students either have better developed only some skills important for mathematics or that they are taught to further develop only some skills in mathematics. Also, this task showed that although the results (related to the basic knowledge of the concept) were given to them in graphic form, many of them did not appreciate it (Student 13 and Student 17). As shown in Table 7, this task got the best results, 33 % of students correctly solved this task while only 36 % of participants correctly answered the question. The overall results showed that only one student in this group was scored 6 of 6.

Table 7. Results obtained for Task 3.

			Knowledge 1		**Skill 1**	
	Frequency	(%)	Frequency	(%)	Frequency	(%)
male	68	61 %	30	44 %	26	38 %
female	43	39 %	11	26 %	11	26 %
gymnasium	100	90 %	36	36 %	31	31 %
TVS	11	10 %	4	36 %	6	55 %
TOTAL	**111**		**40**	**36 %**	**37**	**33 %**

4. Conclusion

Over the last twenty year, the teaching and learning of geometry has changed dramatically under the strong influence of new technology, especially in relation to spatial reasoning and visualization (Kovačević, 2017; Kuzle & Glasnović Gracin, 2020). This turbulent recent history of geometry is not surprising because ever since the 18th century, along with the strengthening of technological progress, the interest in achieving different goals in everyday life also began to grow worldwide, resulting in the same diversity of focuses in mathematical science (and consequently education as well) (Herbst et al., 2017). This diversity naturally led to a significant discourse in the teaching (and learning) of geometry around the world.

In this paper, the focus was on the importance of strengthening the didactical support in connection to the use of construction in relation to learner-generated drawings in mathematics, because this process strongly supports the learning of geometric concepts. Largely due to lack of time, this process is often ignored by many teachers at lower educational levels (or replaced by a faster sketching process that does not have the same didactic value). Unfortunately, today, due to the simplification of the construction process through the use of computers and the earlier introduction of the content of spatial geometry into the educational system in many countries, construction problems are mistakenly put in the background. The problem of the use of the construction in mathematics is closely related to the problem of the use of visual register and requires further interdisciplinary research. However, viewing construction as a mathematical activity means creating ideal objects. Either by folding paper or drawing on a paper (or computer screen), real objects are made, created, and represented as well. Constructing as a fundamental activity in the teaching of geometry should have stronger didactical support, regardless of whether it is constructions with compass and ruler, or dynamic geometric software.

Both sketching and constructing have their place and role in classrooms. Although the students put too much emphasis on the finished drawing when doing construction tasks (and spend too much time on using provided tools), the teacher should be the one to direct the process and focuses on the description of the construction, the verbalization of the process, i.e. should decide which of the two possibilities is more suitable for achieving the given learning outcome through a specific task. As is shown in short quantitative study in connection to three simple geometric concepts, constantly avoiding the use of construction tasks (which are mainly focused on working with one single class-representative) can also have consequences not only in regard to concept development but also in the development of various mathematical competencies.

In the current era of Industry 4.0 visual representation seems to play a crucial role in gaining knowledge mainly because most of the youth prefer graphic (a video or pictorial representation) over textual or symbolic representation. It is probably due to their constant exposure to multimedia environments. Nevertheless, one should not neglect the known cognitive fact that most people can learn more deeply from words and pictures than from words alone so modern education requires graphic representation and it should be implemented in learning all subjects, including math, at all educational levels. Hence, in order to face the problem of proper graphical representation in STEM disciplines, numerous researchers today re-promote the use of learner-generated drawing by suggesting its simultaneous use in addition with reading, writing, and speaking activities.

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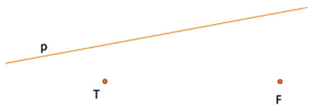
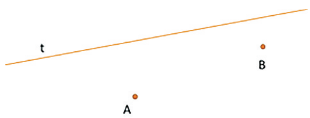
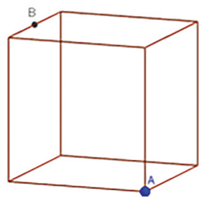
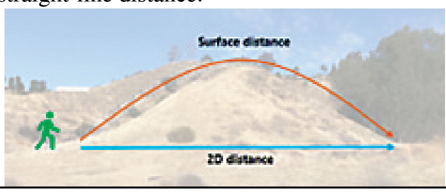
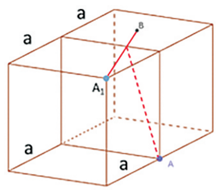
Appendix A

Many traditional geometric construction tasks can be used as a starting point for mathematical modelling tasks where further discussion can be supported by varying some of the parameters involved in solving, or by adding acquisition of more complex skills combining construction task with the tools of analytical geometry.

<table>
<tr><th>Traditional constructive task</th><th>Modelling task</th></tr>
<tr><td>Example 1. A line p and two points A, B are given. Find a point X on the line p so that the sum of the distances $|AX| + |XB|$ is minimal.

</td><td>M-Example 1. The fire truck is nearby a fire that broke out. Unfortunately, the vehicle is without water, so first it has to stop by the river to fill the tank. If a river flow is a straight line in the view from above, and both fire truck and fire are on the same river-side (e.g. two points T, F), constructively determine the fastest path of the fire truck and explain your result.</td></tr>
<tr><td>Example 2. A line p and two points A, B are given (see Figure below). Determine a point X on the line p that is equidistant from points A and B.

</td><td>M-Example 2. Through the suburban areas, it is agreed to introduce a new bus line along the given route t. In the figure below the position of two nearby settlements A and B are given by two points A and B. Constructively determine a position of a new bus stop, point X, on the route t equidistant from both settlements and explain your result.</td></tr>
<tr><td>Example 3. Given a cube and two points A and B on it. Determine the shortest:
a) straight-line distance of two points
b) surface-distance of two points.


The surface-distance is an adjustment to the straight-line distance.

</td><td>M-Example 3. On a cube shaped piece of cheese, a hungry and impatient spider is located on a position A and a fly on a position B. Find the shortest route the spider should take to eat his snack. Is there more than one solution? Explain your result.
Calculate the solution if the length of an edge of a cube is 10, and A is on one of the cube floor vertices and B is positioned at the center of the top side of the cube.


Constructive solution hint using parallel projection is given below.</td></tr>
</table>

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Izrada crteža u matematici: Tko? Kada? Kako?

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Sažetak. Učenje kao generativna aktivnost uključuje davanje smisla informacijama koje se trebaju naučiti mentalnom reorganizacijom i integracijom s prethodnim znanjem, čime se pojedincima omogućuje primjena stečenog znanja u novim situacijama. Promovira se u novije vrijeme kroz različite strategije učenja. Jedna od strategija je strategija crtanja koja uključuje vještinu konstruiranja koja se razvija i u nastavi matematike.

Rad predstavlja rezultate inicijalne provjere znanja provedene na Građevinskom fakultetu Sveučilišta u Zagrebu u listopadu 2022. godine s ciljem stjecanja uvida u razvijenost vještine konstruiranja kao matematičke aktivnosti. Pod pojmom konstrukcija u ovom radu podrazumijeva se matematička aktivnost koja se izvodi u umu s idealnim objektima i obuhvaća aktivnost pronalaženja slijeda koraka konstrukcije uvažavajući svojstva različitih geometrijskih koncepata. Sama realizacija matematičke ideje provodi se kroz crtanje koje se u ovom istraživanju provodi uz dopuštena pomagala šestar i ravnalo (iako se općenito se u nastavi matematike crtanje može izvodi i uz pomoć drugih pomagala).

U radu se analiziraju rezultati konstrukcija 111 studenata prve godine prijediplomskog studija. Matematičko razvijanje vještine konstruiranja od posebnog je značaja za studente tehničkih područja jer se prostorne problemske situacije iz struke rješavaju primjenom konstruktivnih postupaka kroz produbljivanje i proširivanje postojećih geometrijskih znanja i pojmova o prostoru iz osnovne i srednje škole. Uz raspravu o rezultatima provjere koji upućuju na to da postoje izrazite manjkavosti u usvojenosti temeljnih koncepata iz geometrije ravnine na razini visokog obrazovanja u Hrvatskoj, dani su prijedlozi za izbor i dizajn konstrukcijskih zadataka iz područja geometrije prostora koji promiču učenje i matematičko modeliranje kroz upotrebu preciznih matematičkih obrazloženja za konstruktivne postupke koji se provode. Važno je naglasiti da je sposobnost preciznog i samostalnog izvođenja geometrijskih konstrukcija, čak i u doba sve veće digitalizacije, posebno važna za razvoj prostornog rasuđivanja i ne treba ju podcjenjivati.

Ključne riječi: geometrijsko rasuđivanje, crtanje, reprezentacija, matematičko modeliranje, vizualizacija

On Kárteszi Points of a Triangle, Via Three Reflections Theorem and Geometric Algebra

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To memory of my professor and doctor father, Ferenc Kárteszi (1907 – 1989).

Abstract. Let us recall a well-known school task: *In the* (Euclidean $\mathbf{E}^2$) *plane of a triangle ABC we draw regular triangles outward on sides of ABC, say $AB\overline{C}$, $BC\overline{A}$, $CA\overline{B}$, respectively. Prove that the segments $A\overline{A}$, $B\overline{B}$, $C\overline{C}$ intersect each other in a point K, that is the isogonal point of ABC (also called Fermat point or Fermat-Torricelli point) and the distance sum $AK + BK + CK$ is minimal for K among all points of the plane*.

Professor Kárteszi noticed that instead of regular triangles we can draw isosceles ones with all equal base angles, and the above K (called *Kárteszi point*) exists also in the Bolyai-Lobachevsky hyperbolic plane $\mathbf{H}^2$ (in the sphere $\mathbf{S}^2$ as well, (see also Kálmán, 1989 and Sect. 2), the orthocentre, barycentre are specific cases. There is a more general extremum problem of (Yaglom, 1968, problem **83**, with modified notation):

In the plane ($\mathbf{E}^2$) *of a given triangle ABC find a point K such that the quantity $\alpha KA + \beta KB + \gamma KC$, where α, β, γ are given positive numbers, has the smallest possible value.*

This problem leads to a more general triangle configuration and to an analogous extremal point K. Moreover, as a new result of this paper, an extension onto "absolute plane" ($\mathbf{S}^2$, $\mathbf{E}^2$, $\mathbf{H}^2$, $\mathbf{M}^2$ *Minkowski plane*, $\mathbf{G}^2$ *Galilei (or isotropic) plane*) can be formulated and solved by three reflections theorem (see e.g. Molnár, 1978 and Sect. 4, Weiss, 2018), and geometric (Grassmann-Clifford type) *algebra* (Perwass & Hildenbrand, 2004 and Sect. 3). Open problems arise as well. By this we want to follow F. Kárteszi's didactical credo (see also his wonderful book (Kárteszi, 1976) of great international success):

Start with a natural, elementary, visually well understandable task! Then follow the manipulations, tools, new mathematical concepts, the technical machinery; then the solution, occasional theory, further applications, extensions. . .

Keywords: triangle geometry, three reflections theorem, geometric (Grassmann-Clifford type) algebra, absolute geometry, problem solving by similarity transformation

1. Introduction

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died on May 9, 1989.

We start with the solution of the school task in the abstract in Fig. 1. Just as proof with picture without words, as Professor Kárteszi made many times in his seminar in 1970's years in our János Bolyai College (that time Budapest, VIII. Rákóczi str. 5).

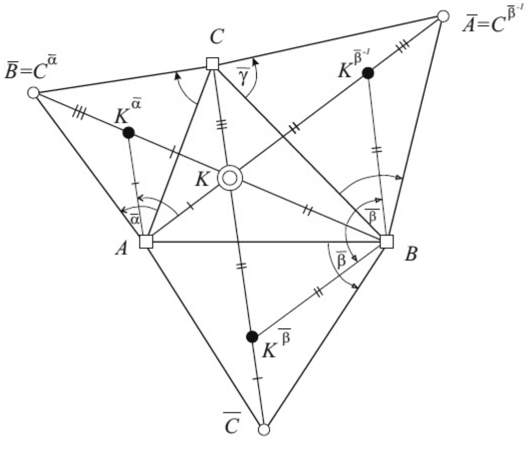

Figure 1. $(\alpha = \beta = \gamma = 1$ in Fig. 2-3$)$: K is the isogonal point of triangle ABC with minimal distance sum $KA + KB + KC = A\overline{A} = B\overline{B} = C\overline{C}$.

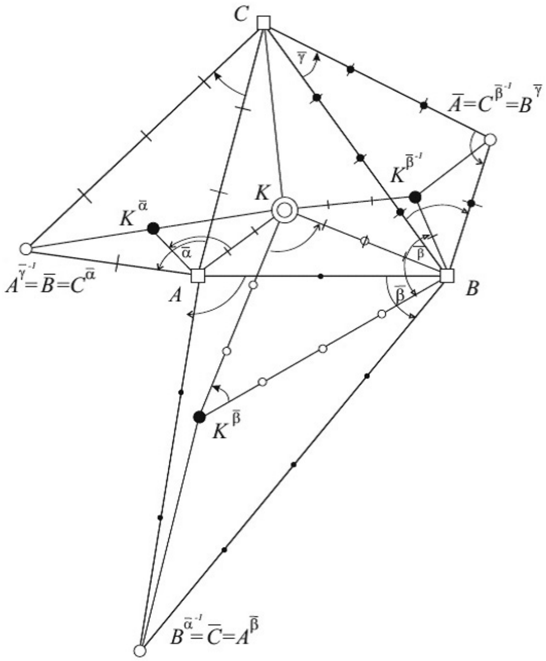

Figure 2. Segmentation (then linearization) of $\alpha KA + \beta KB + \gamma KC$ *by rotatory* similarities $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ about A, B, C, also with angles $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$, respectively.

Then professor Kárteszi turned to his generalisation with isosceles triangles outward on AB, BC, CA of all equal base angles, as in the abstract. He recalled the absolute sine theorem of János Bolyai, see Sect. 2, that guarantees the existence of K as intersection point of $A\overline{A}$, $B\overline{B}$, $C\overline{C}$ (see Kálmán, 1989, who was his doctor student as well). That time he left to me to find the "most general situation". Then I met the above problem of (Yaglom, 1968, having had translated his book volumes I-II and the others onto Hungarian). That problem in Fig. 2-3 led a more general (Euclidean $\mathbf{E}^2$) configuration by similarity transform. I could find a (or the) generalisation (only recently after 50 years) as follows in Fig. 4 and the solution in absolute (projective metric) plane or sphere in Section 4.

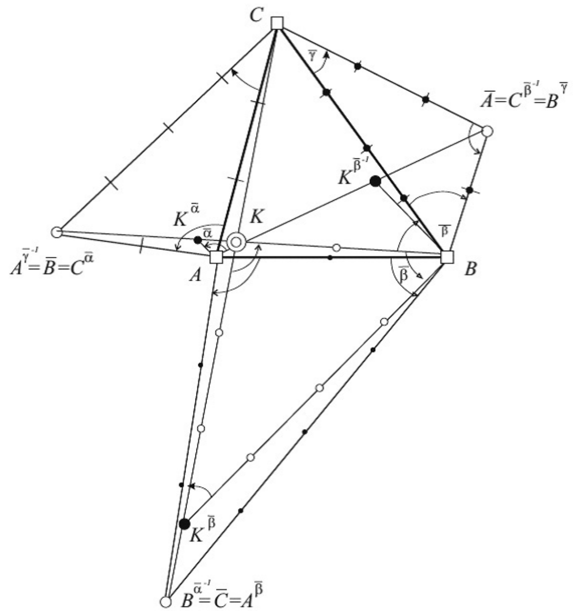

Figure 3. The construction of the extremal point $K = A\overline{A} \cap C\overline{C} \cap B\overline{B}$ of Yaglom's problem in Fig. 2. Here $\alpha : \beta : \gamma \sim 4 : 3 : 2$ (satisfy triangle inequalities), and $\alpha A\overline{A} = \beta B\overline{B} = \gamma C\overline{C}$.

But first, **analyse the problem in Fig. 2.** We can write the (equivalent) proportionality $\alpha KA+\beta KB+\gamma KC \sim KA+(\beta/\alpha)KB+\gamma/\alpha)KC$. Then after AK we can copy $(\beta/\alpha)KB = KK^{\overline{\beta}-1}$ by rotatory similarity $\overline{\beta}^{-1}$ about B with proportion $\gamma/\alpha = 1/2$ (in our concrete choice), so that $BK : KK^{\overline{\beta}-1} : K^{\overline{\beta}-1}B \sim \alpha : \beta : \gamma \sim 4 : 3 : 2$. This rotatory similarity $\overline{\beta}^{-1}$ maps C onto $\overline{A} = C^{\overline{\beta}-1}$, so that $BC : C\overline{A} : \overline{A}B \sim \alpha : \beta : \gamma \sim 4 : 3 : 2$ and $K^{\overline{\beta}-1}C^{\overline{\beta}-1} : KC = \gamma/\alpha = 1/2$. Here BC and $\alpha : \beta : \gamma \sim 4 : 3 : 2$ uniquely determines $\overline{A} = C^{\overline{\beta}-1}$, and $AK + KK^{\overline{\beta}-1} + K^{\overline{\beta}-1}C^{\overline{\beta}-1} \sim AK+(\beta/\alpha)KB+(\gamma/\alpha)KC$, as promised and called *segmentation*. Thus $aKA + \beta KB + \gamma KC \sim KA + (\beta/\alpha)KB + (\gamma/\alpha)KC$ will be as small as possible, iff K lies in the segment $A\overline{A}$ (as linearization), and also in $C\overline{C}$, $B\overline{B}$ as follows.

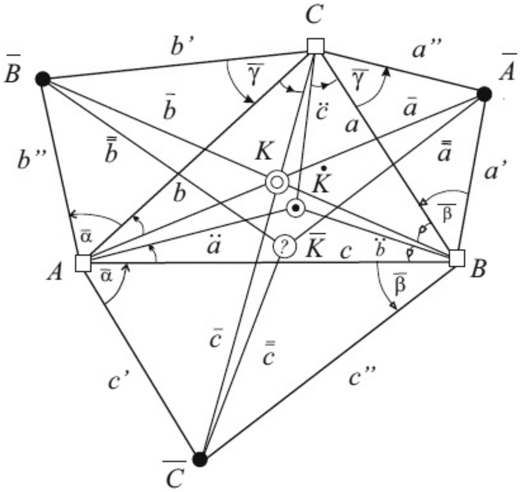

Figure 4. **Our key configuration in the absolute plane**. Let ABC be given triangle (say, of acute angles). Draw, outward on the sides of ABC, angles $\overline{\alpha}$, $-\overline{\alpha}$; $\overline{\beta}$, $-\overline{\beta}$; $\overline{\gamma}$, $-\overline{\gamma}$ at vertices A, B, C, respectively. Then we get triangles $AB\overline{C}$, $BC\overline{A}$, $CA\overline{B}$ (with occasional ideal points $\overline{A}$, $\overline{B}$, $\overline{C}$). Then $A\overline{A} = \overline{a}$, $B\overline{B} = \overline{b}$, $C\overline{C} = \overline{c}$ intersect each other in the Kárteszi point K. Moreover, we see the construction of its focal pair:
$\ddot{a} \cap \ddot{b} \cap \ddot{c} =: K^* \sim (?)\overline{K} := \overline{\overline{a}} \cap \overline{\overline{b}} \cap \overline{\overline{c}}$ (? if $\overline{K}$ exists), furthermore, lines of two surprising line conics a, b, c and a', a'', b', b'', c', c'', respectively, which are confocal ones in specific cases (see Molnár 1978 and Sect. 4).

The same argument holds for starting with C and B instead of A as above. In Fig. 2 we analysed these cases by logical symmetry. Then $BC : C\overline{A} : \overline{A}B \sim \alpha : \beta : \gamma \sim 4 : 3 : 2 \sim \overline{B}C : CA : A\overline{B} \sim B\overline{C} : \overline{C}A : AB$ determine the triangle configuration, where the rotatory similarities and inverses $\overline{\beta}^{-1}$, $\overline{\gamma}$ then $\overline{\beta}$, $\overline{\alpha}^{-1}$ then $\overline{\alpha}$, $\overline{\gamma}^{-1}$ play important roles. These involve the symmetric angles $\overline{\alpha}$, $-\overline{\alpha}$ at vertex A; $\overline{\beta}$, $-\overline{\beta}$ at B; $\overline{\gamma}$, $-\overline{\gamma}$ at C, referred to *three reflections theorem* later on (but e.g. $K^{\overline{\beta}-1} \neq K^{\overline{\gamma}}$!). At Fig. 3 we look the construction: K is the common intersection point of segments $A\overline{A}$, $B\overline{B}$, $C\overline{C}$. Indirect arguments show that this K makes the quantity $\alpha KA + \beta KB + \gamma KC$ as small as possible, from among the points of $(\mathbf{E}^2)$ plane of the given triangle ABC and for given positive real numbers α, β, γ. Moreover, we get $\alpha A\overline{A} = \beta B\overline{B} = \gamma C\overline{C} \sim \alpha KA + \beta KB + \gamma KC$ for this minimum in Fig. 3.

But we have now assumed that α, β, γ satisfy the triangle inequalities. The remaining question is: *What is the solution (situation) in opposite cases?*

2. Application of the absolute sine theorem

János Bolyai's absolute geometry (just 200 years old by his famous Temesvár letter to his father Farkas (Wolfgang) Bolyai, November 3, 1823) has a key of trigonometry, the Absolute sine theorem:

For the triangle ABC of angles α, β, γ at A, B, C and sides $BC = a$, $CA = b$, $AB = c$, respectively, it holds

$$\sin\alpha/\mathrm{o}a = \sin\beta/\mathrm{o}b = \sin\gamma/\mathrm{o}c, \tag{2.1}$$

where e.g. oa *denotes the perimeter of the circle of radius a.*

We remark, that in $\mathbf{E}^2$ holds $\mathrm{o}a = 2\pi a$, in $\mathbf{S}^2$ holds $\mathrm{o}a = 2\pi r\sin(a/r)$, in $\mathbf{H}^2$ holds $\mathrm{o}a = 2\pi k\,\mathrm{sh}(a/k)$; where r is the radius of the sphere as a natural constant; k is a universal constant of $\mathbf{H}^2$, characteristic as the natural distance unit (with $k = \sqrt{-1/K}$, if K denotes the negative sectional curvature, analogous to the spherical positive curvature).

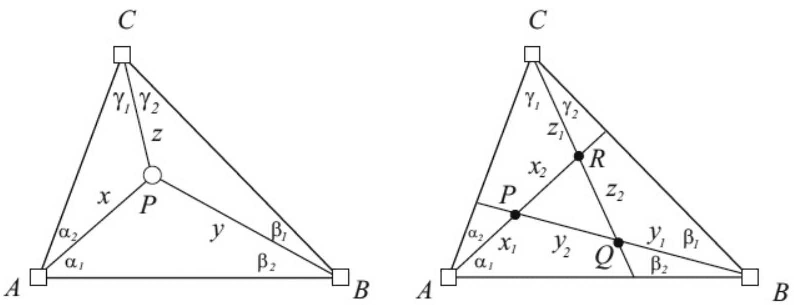

Figure 5. a–b A Ceva type theorem of the absolute geometry

First we recall a Ceva type theorem (Kálmán, 1989, task **199**, Fig. 5. a–b): *The three lines through the vertices of a triangle ABC (say of acute angles for simplicity) intersect each other in an interior point P, if and only if (or iff, with notations of Fig. 5)*

$$\sin\alpha_1\sin\beta_1\sin\gamma_1/\sin\alpha_2\sin\beta_2\sin\gamma_2 = 1. \tag{2.2}$$

Proof. *a) If P exists:* We write the sine theorem for ABP, BCP and CAP, respectively:

$$\sin\alpha_1/\sin\beta_2 = \mathrm{o}y/\mathrm{o}x, \sin\beta_1/\sin\gamma_2 = \mathrm{o}z/\mathrm{o}y, \sin\gamma_1/\sin\alpha_2 = \mathrm{o}x/\mathrm{o}z.$$

Multiplying the left and right sides, respectively, we get (2.2).

b) Indirectly, we assume (2.2) and Fig. 5.b: Again, by sine theorem for ABP it holds

$$\mathrm{o}(y_1 + y_2)/\mathrm{o}x_1 = \sin\alpha_1/\sin\beta_2, \text{ etc. for } BCQ \text{ and } CAR \text{ we get by (2.2)}$$

$$\mathrm{o}(y_1 + y_2)\mathrm{o}(z_1 + z_2)\mathrm{o}(x_1 + x_2)/x_1y_1z_1 = 1$$

that would lead to a contradiction, if $x_2 = y_2 = z_2 = 0$, i.e. $P = Q = R$ would not be true. □

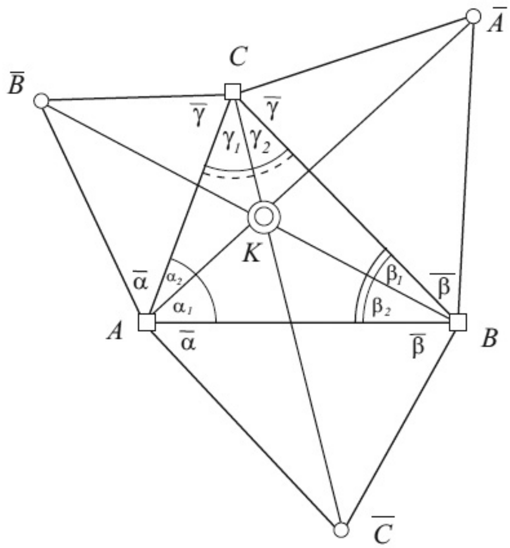

Figure 6. Triangle configuration for Kárteszi point K in absolute geometry, compare also with Fig. 1–4.

Then comes the **Kárteszi point theorem** (in Fig. 6, task **200** in Kálmán, 1989) as an "easy" consequence:

Proof. We write the sine theorem for triangles $A\overline{A}B$ and $A\overline{A}C$, respectively:

$$\sin\alpha_1/\sin(\beta + \overline{\beta}) = \mathrm{o}B\overline{A}/\mathrm{o}A\overline{A} \text{ and } \sin\alpha_2/\sin(\gamma + \overline{\gamma}) = \mathrm{o}C\overline{A}/\mathrm{o}A\overline{A}$$

$$\textit{(imply)} \Rightarrow \sin\alpha_1\sin(\gamma + \overline{\gamma})/\sin\alpha_2\sin(\beta + \overline{\beta}) = \mathrm{o}B\overline{A}/\mathrm{o}C\overline{A} = \sin\overline{\gamma}/\sin\overline{\beta}$$

and other two analogous implications by logical symmetry. Multiplying the corresponding sides, we get again by simplifications $\sin\alpha_1\sin\beta_1\sin\gamma_1/\sin\alpha_2\sin\beta_2\sin\gamma_2 = 1$ the Ceva type equality (2.2), thus

$$A\overline{A} \cap B\overline{B} \cap C\overline{C} = K \tag{2.3}$$

is just the Kárteszi point. □

It arises **the problem**: What are the connections among $A\overline{A}$, $B\overline{B}$, $C\overline{C}$ and data of ABC, and $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$? (Compare with the Euclidean situations!)

3. Projective metric plane over a real vector space $\mathbf{V}^3$ and its dual $\mathbf{V}_3$. Geometric (Grassmann-Clifford type) algebra

For our description we need standard but lengthy series of concepts, notations, so we try to apply the (hopefully) informative Figs. 7.*a*–*c* below.

Fig. 7.*a* shows our scene, how the real affine (Euclidean) plane A^2 is embedded into the affine (Euclidean) space $A^3(O, \mathbf{V}^3, \boldsymbol{V}_3)$, into a projective plane $P^2 A^2 \cup i)$, furthermore into the projective sphere $P\mathbf{S}^2$ that can also be considered a double affine plane extended by a double ideal line i at infinity ∞. Here $\mathbf{V}^3$ is a (say) left vector space over real ($\mathbf{R}$) number field, $\boldsymbol{V}_3$ is its dual linear (then right) form space. $\mathbf{V}^3 \ni \mathbf{x} = x^i \mathbf{e}_i$ with a basis $\{\mathbf{e}_i\}$ $(i = 0, 1, 2)$, $\boldsymbol{V}_3 \ni \boldsymbol{u} = \boldsymbol{e}^j u_j$ with dual basis $\{\boldsymbol{e}^j\}$ defined by $\mathbf{e}_i \boldsymbol{e}^j = \delta_i^j$ (the Kronecker delta symbol; Einstein-Schouten index conventions will also be used; of course, null vector $\mathbf{o}$ and null form $\boldsymbol{o}$ are excluded).

At Fig 7.*b* we look also a triangle coordinate system $A_0A_1A_2A$. Out of A^2 we take an origin O, from that vectors point to A^2 so P^2 and $P\mathbf{S}^2$ up to $\sim$ proportionalities below, respectively. That means vectors $\mathbf{x} \sim c\mathbf{x}$ describe the same point X $(\mathbf{x})$ of $P\mathbf{S}^2$ iff $0 < c \in \mathbf{R}$, and $(\mathbf{x}) = (-\mathbf{x})$ for P^2. Linear forms of $\boldsymbol{V}_3$ describe the planes through the origin O of A^3, the proportionality classes describe the lines of P^2 and or (main cirles of) $P\mathbf{S}^2$. A point A $(OA = \mathbf{a} = \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2$, called unit point) fixes the coordinate system $A_0A_1A_2A$ (up to proportionality). Then lines $b^0 = A_1A_2$, $b^1 = A_2A_0$, $b^2 = A_0A_1$ and unit line $b(\boldsymbol{b} = \boldsymbol{b}^0 + \boldsymbol{b}^1 + \boldsymbol{b}^2)$, the dual coordinate system in $\mathbf{V}^3$ will also be fixed.

At Fig. 7.*c* we look the projective metric Beltrami-Cayley-Klein (B-C-K) model of the Bolyai-Lobachevsky hyperbolic ($\mathbf{H}^2$) plane defined by a polar line $(u) \rightarrow$ pole point (U) *polarity*$(\heartsuit)$: $\boldsymbol{V}_3 \rightarrow V^3$, $u(\mathbf{u}) \rightarrow U(\boldsymbol{u}\heartsuit) = U(\mathbf{u})$ and its quadratic conic of signature $(-, +, +)$. This is equivalently given by a symmetric bilinear scalar product $\langle\, ,\rangle : \boldsymbol{V}_3 \times V_3 \rightarrow \mathbf{R}$, $\langle \boldsymbol{u} = \boldsymbol{b}^i u_i, \boldsymbol{v} = \boldsymbol{b}^j v_j\rangle := (\boldsymbol{v}\heartsuit\boldsymbol{u}) = (\boldsymbol{u}\heartsuit\boldsymbol{v}) = u_i, \pi^{ij}\mathbf{e}_j\boldsymbol{e}^k v_k = u_i, \pi^{ij}\delta_j^k v_k = u_i, \pi^{ij} v_j$ with symmetry $\pi^{ij} = \pi^{ji}$. In the usual Euclidean (B-C-K) model, where the polarity is invertible, $(\heartsuit)^{-1} := (\spadesuit) : \mathbf{V}^3 \rightarrow \boldsymbol{V}_3$, $x \rightarrow x^{\spadesuit} = \boldsymbol{x}$. Distance (d) and angle (a) metric can be defined by the scalar product:

$$\mathrm{ch}[d(\mathbf{x}, \mathbf{y})/k] = -\langle \mathbf{x}, \mathbf{y}\rangle / \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle \langle \mathbf{y}, \mathbf{y}\rangle}, \ \cos[a(\boldsymbol{u}, \boldsymbol{v})] = -\langle \boldsymbol{u}, v\rangle / \sqrt{\langle \boldsymbol{u}, u\rangle \langle \boldsymbol{v}, \boldsymbol{v}\rangle},$$

and this can be extended to complex distance and angles, as usual by complex exponential, then ch and cos functions.

$$0 = -x^0x^0 + x^1x^1 + x^2x^2 \quad \text{and} \quad 0 = -u_0u_0 + u_1u_1 + u_2u_2$$

describe the points and the lines of absolute conic, respectively, as a unit circle and its touching lines in homogeneous coordinates. The elliptic (spherical $\mathbf{S}^2$) plane is characterized by signature $(+, +, +)$, the Euclidean ($\mathbf{E}^2$) plane by $(0, +, +)$, the Minkowski ($\mathbf{M}^2$) plane by $(0, +, -)$, the Galilei ($\mathbf{G}^2$ or isotropic) plane by $(0, 0, +)$, no more detailed.

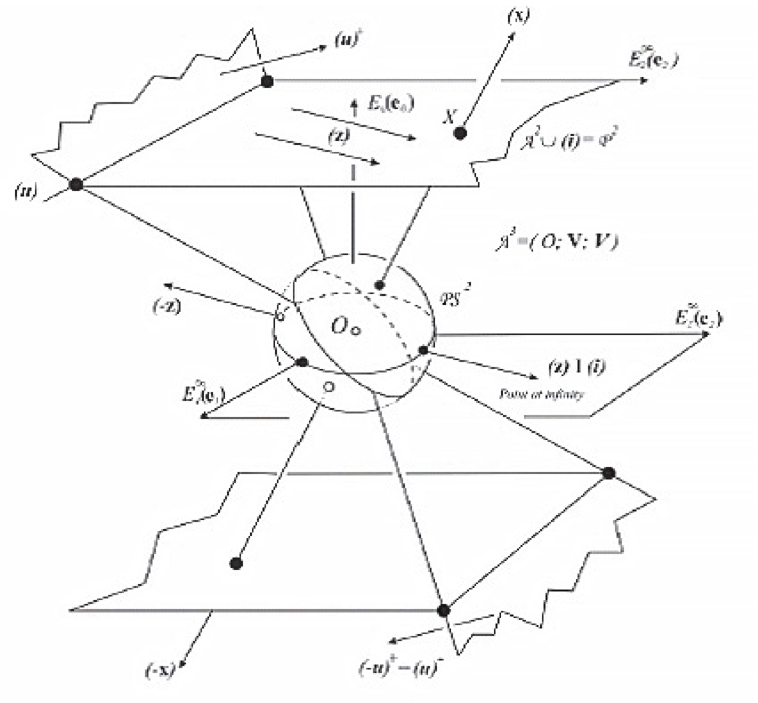

Figure 7.a The real ($\mathbf{R}$) affine (Euclidean) plane A^2 embedded into the affine (Euclidean) space. $A^3(O, \mathbf{V}^3, \boldsymbol{V}_3)$, into a projective plane $P^2(A^2 \cup i)$, furthermore into the projective sphere $P\mathbf{S}^2$.

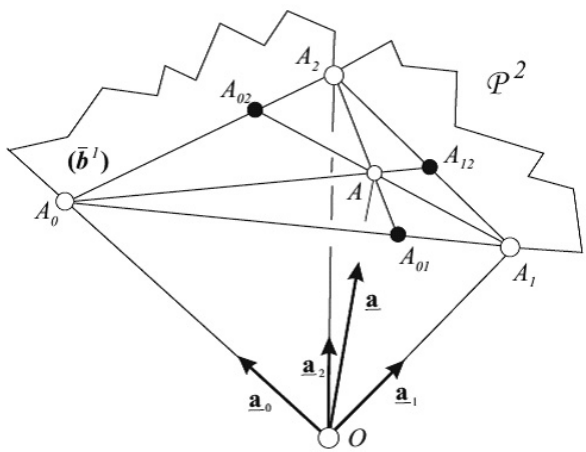

Figure 7.b The triangle coordinate system $A_0A_1A_2A$ for points and lines.

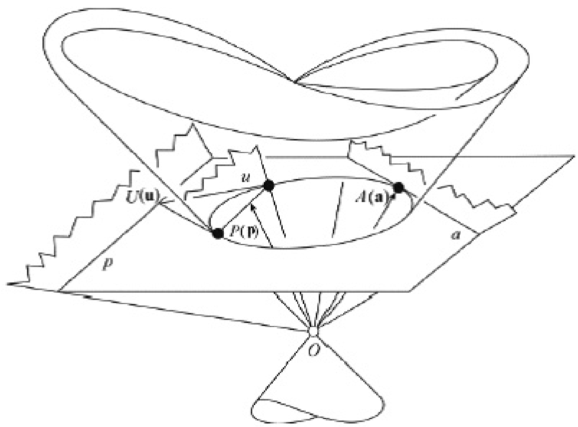

Figure 7.c The Beltrami-Cayley-Klein (B-C-K) model of Bolyai-Lobachevsky hyperbolic ($\mathbf{H}^2$) plane with its absolute conic and asymptote cone with the two sheets hyperboloid in the embedding 3-space $A^3(O, \mathbf{V}^3, \boldsymbol{V}_3)$, given by the linear symmetric *absolute* polar line $(u) \to$ pole point (U) *polarity* $(\blacktriangledown) : \boldsymbol{V}_3 \to \mathbf{V}^3$, $u(\mathbf{u}) \to U(\boldsymbol{u}\blacktriangledown) = U(\mathbf{u})$ or symmetric bilinear scalar product $\langle \boldsymbol{u} = \boldsymbol{b}^i u_i, \boldsymbol{v} = \boldsymbol{b}^j v_j \rangle = u_i \pi^{ij} v_j$, $\pi^{ij} = \pi^{ji}$.

A typical projective (linear) mapping is the central axial collineation with centre $C(\mathbf{c})$, axis $a(\boldsymbol{a})$ and parameter p for points $X(\mathbf{x})$ and lines $u(\boldsymbol{u})$, respectively:

$$\sigma_a^C : \mathbf{V}^3 \to \mathbf{V}^3, \quad X(\mathbf{x}) \to Y(\mathbf{y}), \quad \mathbf{y} = \mathbf{x} - p(\mathbf{x}\boldsymbol{a})\mathbf{c};$$
$$\boldsymbol{V}_3 \to \boldsymbol{V}_3, \quad u(\boldsymbol{u}) \to v(\boldsymbol{v}), \quad \boldsymbol{v} = \boldsymbol{u} - \boldsymbol{a}(\mathbf{c}\boldsymbol{u}),$$

where

$$p = 1/[(\mathbf{c}\boldsymbol{a}) - 1]. \tag{3.1}$$

This will be an involutive (involutory) reflection σ_a^A for polar axis line $a(\boldsymbol{a})$ and pole centre point $A(\mathbf{a})$ as follows

$$\sigma_a^A : \mathbf{V}^3 \to \mathbf{V}^3,\ X(\mathbf{x}) \to Y(\mathbf{y}),\ \mathbf{y} = \mathbf{x} - [2(\mathbf{x}\boldsymbol{a})/(\mathbf{a}\boldsymbol{a})]\mathbf{a};$$

$$\boldsymbol{V}_3 \to \boldsymbol{V}_3,\ u(\boldsymbol{u}) \to v(\boldsymbol{v}),\ \boldsymbol{v} = \boldsymbol{u} - \boldsymbol{a}[(\mathbf{a}\boldsymbol{u})2/(\mathbf{a}\boldsymbol{a})],\ \text{if}\ (\mathbf{a}\boldsymbol{a}) \neq 0. \tag{3.2}$$

Thus we can define general projective (linear) mappings as products of central axial collineations (as inverse pairs, acting on vectors from right, then on forms from left). Congruent transforms of projective metric planes will be the products of (say) line reflections.

Then we can use also the *reflection geometry* of F. Bachmann (see e.g. Molnár 1978), as indicated later on.

The **Grassmann-Clifford type geometric algebra** will be defined in the most elementary form in analogy of introducing alternating bilinear cross product of classical Euclidean space geometry (but now in projective plane version): point $\times$ point $=$ their-connecting-line and line $\times$ line $=$ their-intersection-point. Of course, these will be by the above projective metric 3-vector-form calculus or rather 3-form-vector calculus for projective metric plane geometry, i.e. for $\mathbf{S}^2$, $\mathbf{H}^2$ mostly, but also for $\mathbf{E}^2$, $\mathbf{M}^2$, $\mathbf{G}^2$ with degenerate polarities, not detailed more. Let the multilinear usual alternating determinant function defined and normed on a dual basis pair $\{\mathbf{e}_i\}$, $\{\boldsymbol{e}^j\}$ $(i,j=0,1,2)$ as $\det(\mathbf{e}_0,\mathbf{e}_1,\mathbf{e}_2)=1=\det(\boldsymbol{e}^0,\boldsymbol{e}^1,\boldsymbol{e}^2)$

$$X(\mathbf{x})\times Y(\mathbf{y})=u(u) \text{ by } (\mathbf{z}u)=\det(\mathbf{x},\mathbf{y},\mathbf{z})=0,$$

u is the linear form (directed line of $P(\mathbf{S}^2)$) on the variable point $Z(\mathbf{z})$.

$$u(\boldsymbol{u})\times v(\boldsymbol{v})=X(\mathbf{x}) \text{ by } (\mathbf{x}\boldsymbol{w})=\det(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=0, \tag{3.3}$$

$\mathbf{x}$ is the vector (its class in $P(\mathbf{S}^2)$) on the variable (directed) line $w(\boldsymbol{w})$.

We can check some important identities and duals (as well known for usual cross product) in

Theorem 3.
1.a.

$$\mathbf{p}=:(\mathbf{x}\ \times\mathbf{y})\times\boldsymbol{u}=(\mathbf{y}\boldsymbol{u})\mathbf{x}-(\mathbf{x}\boldsymbol{u})\mathbf{y},\ \boldsymbol{s}=:(\boldsymbol{u}\times\boldsymbol{v})\times\mathbf{x}\ =\boldsymbol{u}(\mathbf{x}\boldsymbol{v})-\boldsymbol{v}(\mathbf{x}\boldsymbol{u}). \tag{3.4}$$

1.b. *The linear form* $\mathbf{x}\ \times\mathbf{y}$ *takes on the vector* $\boldsymbol{u}\ \times\boldsymbol{v}$ *the real number*

$$(\boldsymbol{u}\times\boldsymbol{v},\mathbf{x}\times\mathbf{y})=\det\begin{vmatrix}\mathbf{x}\boldsymbol{u} & \mathbf{x}\boldsymbol{v}\\ \mathbf{y}\boldsymbol{u} & \mathbf{y}\boldsymbol{v}\end{vmatrix}=(\mathbf{x}\boldsymbol{u})(\mathbf{y}\boldsymbol{v})-(\mathbf{x}\boldsymbol{v})(\mathbf{y}\boldsymbol{u}). \tag{3.5}$$

1.c. *The linear form* $\boldsymbol{u}\ =\mathbf{x}\ \times\mathbf{y}$ *takes the real value* $\det(\mathbf{z},\mathbf{x},\mathbf{y})$ *on the vector* $\mathbf{z}$.

1.d. *The pole of* $\boldsymbol{u}\ =\ \mathbf{x}\times\mathbf{y}$ *is* $\mathbf{u}\ =\ \boldsymbol{u}^{\heartsuit}\ =\ \mathbf{x}^{\spadesuit}\times\mathbf{y}^{\spadesuit}$*, if* $(\spadesuit)$*, the inverse polarity of* $(\heartsuit)$ *exists* $(\mathbf{S}^2,\mathbf{H}^2)$. *Then the scalar product, distance and angle can also be extended.*

As an important application of our analytic tools, we cite

Theorem 3.2 (The three reflections theorem for projective metric plane P^2 or sphere $P\mathbf{S}^2$).

Three lines, $a(\boldsymbol{a})$, $b(\boldsymbol{b})$, $c(\boldsymbol{c})$ *coincide with a point* $U(\mathbf{u})$ *all in* P^2 *or* $P\mathbf{S}^2$*, iff there is a fourth line* $d(\boldsymbol{d})$*, such that for the corresponding reflections (with the identical notations) it holds* $\boldsymbol{abc}\ =\boldsymbol{d}$*, iff*

$$\boldsymbol{d}\ =\boldsymbol{a}\langle\boldsymbol{b},\boldsymbol{c}\rangle\ -\boldsymbol{b}\langle\boldsymbol{c},\boldsymbol{a}\rangle\ +\boldsymbol{c}\langle\boldsymbol{a},\boldsymbol{b}\rangle\,; \tag{3.6}$$

or equivalently $\boldsymbol{ab}=\boldsymbol{dc}$ *as a rotation about* $U(\mathbf{u})$*, or (the same) translation along* $u(\boldsymbol{u})$*. Moreover, either* $\boldsymbol{a}=\boldsymbol{b}\beta+\boldsymbol{d}$ *and* $\boldsymbol{c}=\boldsymbol{b}\langle\boldsymbol{d},\boldsymbol{d}\rangle\ +\boldsymbol{d}\langle\boldsymbol{b},\boldsymbol{b}\rangle\,\beta$*, for* $\mathbf{R}\ni\beta\neq0$*; or* $\boldsymbol{a}\ =\ \boldsymbol{b}$ *and* $\boldsymbol{c}\ =\ \boldsymbol{d}$*; or* $\boldsymbol{a}\ =\ \boldsymbol{c}$ *and* $\langle\boldsymbol{d},d\rangle\ =\ \langle\boldsymbol{b},\boldsymbol{b}\rangle\,\beta\beta$ *occur for non-zero real coefficients.*

Sketch of this last proof (because of later application, the former one is similar; the equivalence is also easy). Apply(3.2) for $\boldsymbol{V}_3$, say, express $\boldsymbol{ab} = \boldsymbol{dc}$!

$$\boldsymbol{x} \to x^{ab} = \boldsymbol{x} - \boldsymbol{a}\,[\langle \boldsymbol{a},\boldsymbol{x}\rangle\, 2/\langle \boldsymbol{a},\boldsymbol{a}\rangle\,] - \boldsymbol{b}\,[\langle \boldsymbol{b},\boldsymbol{x}\rangle\, 2/\langle \boldsymbol{b},\boldsymbol{b}\rangle\,]$$

$$+\boldsymbol{b}\,[\langle \boldsymbol{b},\boldsymbol{a}\rangle\,\langle \boldsymbol{a},\boldsymbol{x}\rangle\, 4/\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\langle \boldsymbol{a},\boldsymbol{a}\rangle\,],$$

$$\boldsymbol{x} \to x^{dc} = \boldsymbol{x} - \boldsymbol{d}\,[\langle \boldsymbol{d},\boldsymbol{x}\rangle\, 2/\langle \boldsymbol{d},\boldsymbol{d}\rangle\,] - \boldsymbol{c}[\langle \boldsymbol{c},\boldsymbol{x}\rangle\, 2/\langle \boldsymbol{c},\boldsymbol{c}\rangle\,]$$

$$+\boldsymbol{c}[\langle \boldsymbol{c},\boldsymbol{d}\rangle\,\langle \boldsymbol{d},\boldsymbol{x}\rangle\, 4/\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\langle \boldsymbol{d},\boldsymbol{d}\rangle\,].$$

From the proportionality of the right sides above for every $\boldsymbol{x} \in \boldsymbol{V}_3$ it follows

$$-\boldsymbol{a}\,[\langle \boldsymbol{a},\boldsymbol{x}\rangle\, 2/\langle \boldsymbol{a},\boldsymbol{a}\rangle\,] - \boldsymbol{b}\,[\langle \boldsymbol{b},\boldsymbol{x}\rangle\, 2/\langle \boldsymbol{b},\boldsymbol{b}\rangle\,] + \boldsymbol{b}\,[\langle \boldsymbol{b},\boldsymbol{a}\rangle\,\langle \boldsymbol{a},\boldsymbol{x}\rangle\, 4/\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\langle \boldsymbol{a},\boldsymbol{a}\rangle\,]$$

$$= -\boldsymbol{d}\,[\langle \boldsymbol{d},\boldsymbol{x}\rangle\, 2/\langle \boldsymbol{d},\boldsymbol{d}\rangle\,] - \boldsymbol{c}[\langle \boldsymbol{c},\boldsymbol{x}\rangle\, 2/\langle \boldsymbol{c},\boldsymbol{c}\rangle\,] + \boldsymbol{c}[\langle \boldsymbol{c},\boldsymbol{d}\rangle\,\langle \boldsymbol{d},\boldsymbol{x}\rangle\, 4/\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\langle \boldsymbol{d},\boldsymbol{d}\rangle\,]. \tag{3.7}$$

Then after a step by step procedure, now by distinguished $\boldsymbol{b}$ and $\boldsymbol{d}$ for our reason, it follows

$$\boldsymbol{a} = \boldsymbol{b}\beta + \boldsymbol{d}\delta, \quad \boldsymbol{c} = \boldsymbol{b}\overline{\beta} + \boldsymbol{d}\overline{\delta}, \quad \text{so} \quad \langle \boldsymbol{a},\boldsymbol{x}\rangle\, = \langle \boldsymbol{b},\boldsymbol{x}\rangle\,\beta + \langle \boldsymbol{d},\boldsymbol{x}\rangle\,\delta$$

and

$$\langle \boldsymbol{c},\boldsymbol{x}\rangle\, = \langle \boldsymbol{b},\boldsymbol{x}\rangle\,\overline{\beta} + \langle \boldsymbol{d},\boldsymbol{x}\rangle\,\overline{\delta}. \tag{3.8}$$

By substitution into the above equation (3.7), we get

$$-(\boldsymbol{b}\beta + \boldsymbol{d}\delta)\{[\langle \boldsymbol{b},\boldsymbol{x}\rangle\,\beta 2 + \langle \boldsymbol{d},\boldsymbol{x}\rangle\,\delta 2]\langle \boldsymbol{b},\boldsymbol{b}\rangle\,/\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\}$$

$$-\boldsymbol{b}\,\{[\langle \boldsymbol{b},\boldsymbol{x}\rangle\, 2]\langle \boldsymbol{a},\boldsymbol{a}\rangle\,/\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\}$$

$$+\boldsymbol{b}\{\langle \boldsymbol{b},\boldsymbol{a}\rangle\,[\langle \boldsymbol{b},\boldsymbol{x}\rangle\,\beta + \langle \boldsymbol{d},\boldsymbol{x}\rangle\,\delta]4/\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\}$$

$$= -\boldsymbol{d}\,\{\langle \boldsymbol{d},\boldsymbol{x}\rangle\, 2\langle \boldsymbol{c},\boldsymbol{c}\rangle\,/\langle \boldsymbol{d},\boldsymbol{d}\rangle\,\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\}$$

$$-(\boldsymbol{b}\overline{\beta} + \boldsymbol{d}\overline{\delta})\{[\langle \boldsymbol{b},\boldsymbol{x}\rangle\,\overline{\beta} 2 + \langle \boldsymbol{d},\boldsymbol{x}\rangle\,\overline{\delta} 2]\langle \boldsymbol{d},\boldsymbol{d}\rangle\,/\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\langle \boldsymbol{d},\boldsymbol{d}\rangle\,\}$$

$$+(\boldsymbol{b}\overline{\beta} + \boldsymbol{d}\overline{\delta})[\langle \boldsymbol{c},\boldsymbol{d}\rangle\,\langle \boldsymbol{d},\boldsymbol{x}\rangle\, 4/\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\langle \boldsymbol{d},\boldsymbol{d}\rangle\,]. \tag{3.9}$$

After having collected coefficients of $\boldsymbol{b}$ and $\boldsymbol{d}$, respectively, both have to be zero for every $\boldsymbol{x}$, so we get 4 equations besides (3.8), but only one provides essentially new information:

$$\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\beta\delta} = \langle \boldsymbol{c},\boldsymbol{c}\rangle\,\beta\delta. \tag{3.10}$$

The others are consequences,

$$\text{either} \quad \beta\overline{\delta} = \overline{\beta}\delta \quad \text{or and} \quad \langle \boldsymbol{b},\boldsymbol{b}\rangle\,\beta\overline{\beta} = \langle \boldsymbol{d},\boldsymbol{d}\rangle\,\delta\overline{\delta}.$$

From these and the proportionalities follow the cases of Theorem 3.2 step by step. □

4. Generalized Kárteszi points by geometric algebra

We start with Fig. 4, where our notations are indicated, and apply the above three reflections Theorem 3.2 in the corresponding new roles:

at vertex A:

$$\boldsymbol{c'c} = \boldsymbol{bb''}, \quad \boldsymbol{c'} = \boldsymbol{c}\overline{\alpha} + \boldsymbol{b}, \quad \boldsymbol{b''} = \boldsymbol{c}\langle \boldsymbol{b},\boldsymbol{b}\rangle\ + \boldsymbol{b}\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\overline{\alpha};$$

at B:

$$\boldsymbol{a'a} = \boldsymbol{c}|\boldsymbol{c''}, \quad \boldsymbol{a'} = \boldsymbol{a}\overline{\beta} + \boldsymbol{c}, \quad \boldsymbol{c''} = \boldsymbol{a}\langle \boldsymbol{c},\boldsymbol{c}\rangle\ + \boldsymbol{c}\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\beta};$$

at vertex C:

$$\boldsymbol{b'b} = \boldsymbol{aa''}, \quad \boldsymbol{b'} = \boldsymbol{b}\overline{\gamma} + \boldsymbol{a}, \quad \boldsymbol{a''} = \boldsymbol{b}\langle \boldsymbol{a},\boldsymbol{a}\rangle\ + \boldsymbol{a}\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\gamma}. \tag{4.1}$$

Then, by Theorem 3.1 and logical symmetry for short,

for $\overline{A}(\overline{\mathbf{a}})$:

$$\begin{aligned}\boldsymbol{a'} \times \boldsymbol{a''} &= (\boldsymbol{a}\overline{\beta} + \boldsymbol{c}) \times (\boldsymbol{b}\langle \boldsymbol{a},\boldsymbol{a}\rangle\ + \boldsymbol{a}\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\gamma})\\ &= (\boldsymbol{a}\times\boldsymbol{b})\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\beta} - (\boldsymbol{b}\times\boldsymbol{c})\langle \boldsymbol{a},\boldsymbol{a}\rangle\ + (\boldsymbol{c}\times\boldsymbol{a})\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\gamma};\end{aligned}$$

for $\overline{B}(\overline{\mathbf{b}})$:

$$\boldsymbol{b'} \times \boldsymbol{b''} = (\boldsymbol{b}\times\boldsymbol{c})\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\gamma} - (\boldsymbol{c}\times\boldsymbol{a})\langle \boldsymbol{b},\boldsymbol{b}\rangle\ + (\boldsymbol{a}\times\boldsymbol{b})\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\overline{\alpha};$$

for $\overline{C}(\overline{\mathbf{c}})$:

$$\boldsymbol{c'} \times \boldsymbol{c''} = (\boldsymbol{c}\times\boldsymbol{a})\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\overline{\alpha} - (\boldsymbol{a}\times\boldsymbol{b})\langle \boldsymbol{c},\boldsymbol{c}\rangle\ + (\boldsymbol{b}\times\boldsymbol{c})\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\beta}. \tag{4.2}$$

Then for $A\overline{A}$:

$$\overline{\boldsymbol{a}} \sim \boldsymbol{c}\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\gamma} - \boldsymbol{b}\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\beta};$$

for $B\overline{B}$:

$$\overline{\boldsymbol{b}} \sim \boldsymbol{a}\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\overline{\alpha} - \boldsymbol{c}\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\gamma};$$

for $C\overline{C}$:

$$\overline{\boldsymbol{c}} \sim \boldsymbol{b}\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\beta} - \boldsymbol{a}\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\overline{\alpha};$$

and finally for K:

$$\overline{\boldsymbol{a}}\times\overline{\boldsymbol{b}} \sim \langle \boldsymbol{a},\boldsymbol{a}\rangle\,\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\beta}\overline{\gamma}(\boldsymbol{b}\times\boldsymbol{c}) + \langle \boldsymbol{b},\boldsymbol{b}\rangle\,\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\overline{\gamma}\overline{\alpha}(\boldsymbol{c}\times\boldsymbol{a}) + \langle \boldsymbol{c},\boldsymbol{c}\rangle\,\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\alpha}\overline{\beta}(\boldsymbol{a}\times\boldsymbol{b}).$$

More simply,

$$K \sim \overline{\boldsymbol{a}}\times\overline{\boldsymbol{b}} \sim \mathbf{a}/\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\overline{\alpha} + \mathbf{b}/\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\beta} + \mathbf{c}/\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\gamma} \sim \overline{\boldsymbol{b}}\times\overline{\boldsymbol{c}} \sim \overline{\boldsymbol{c}}\times\overline{\boldsymbol{a}} \in \mathbf{V}^3. \tag{4.3}$$

Theorem 4.1 on Kárteszi point $K = A\overline{A} \cap C\overline{C} \cap B\overline{B}$ by Fig. 4 has already been proved above for projective metric planes (spheres) in more general form! □

But the tree reflections configurations promise more observations by our paper (Molnár, 1978) as a new nice (mystique a little bit).

Closing Theorem 4.2 (Fig. 4). *By three reflections theorem hold the following relations:*

$$\overline{\boldsymbol{a}}\,\overline{\boldsymbol{k}} = \overline{\boldsymbol{b}}\,\overline{\boldsymbol{c}}; \quad \ddot{\boldsymbol{a}}\,\ddot{\boldsymbol{k}} = \ddot{\boldsymbol{b}}\,\ddot{\boldsymbol{c}}; \quad \overline{\overline{\boldsymbol{a}}}\,\overline{\overline{\boldsymbol{k}}} = \overline{\overline{\boldsymbol{b}}}\,\overline{\overline{\boldsymbol{c}}};$$

where

$$\overline{\boldsymbol{a}} \sim \boldsymbol{c}\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\gamma} - \boldsymbol{b}\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\beta}, \quad \overline{\boldsymbol{b}} \sim \boldsymbol{a}\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\overline{\alpha} - \boldsymbol{c}\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\gamma}, \quad \overline{\boldsymbol{c}} \sim \boldsymbol{b}\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\beta} - \boldsymbol{a}\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\overline{\alpha}$$

as above; and

$$\ddot{\boldsymbol{a}} \sim \boldsymbol{c}\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\beta} - \boldsymbol{b}\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\overline{\gamma}, \quad \ddot{\boldsymbol{b}} \sim \boldsymbol{a}\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\gamma} - \boldsymbol{c}\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\alpha}, \quad \ddot{\boldsymbol{c}} \sim \boldsymbol{b}\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\overline{\alpha} - \boldsymbol{a}\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\beta}.$$

Moreover, $\boldsymbol{a}'\,\overline{\boldsymbol{a}}\,\ddot{\boldsymbol{a}} = \overline{\overline{\boldsymbol{a}}}$ *implies by three reflections theorem first*

$$\overline{\overline{\boldsymbol{a}}} \sim \boldsymbol{a}'\langle \boldsymbol{a}'',\boldsymbol{a}''\rangle\,\overline{\beta} - \boldsymbol{a}''\langle \boldsymbol{a}',\boldsymbol{a}'\rangle\,\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\gamma},$$

then two cyclically analogous equations and their explicit consequences come into account.

It turns out that the above Kárteszi point

$$K \sim \overline{\boldsymbol{a}} \times \overline{\boldsymbol{b}} \sim \mathbf{a}/\langle \boldsymbol{c},\boldsymbol{c}\rangle\,\overline{\alpha} + \mathbf{b}/\langle \boldsymbol{a},\boldsymbol{a}\rangle\,\overline{\beta} + \mathbf{c}/\langle \boldsymbol{b},\boldsymbol{b}\rangle\,\overline{\gamma}$$

has a focal pair $K^* \sim \overline{K}$*, where*

$$\ddot{\boldsymbol{a}} \cap \ddot{\boldsymbol{b}} \cap \ddot{\boldsymbol{c}} =: K^* \sim (\overline{\alpha}/\langle \boldsymbol{b},\boldsymbol{b}\rangle)\boldsymbol{a} + (\overline{\beta}/\langle \boldsymbol{c},\boldsymbol{c}\rangle)\boldsymbol{b} + (\overline{\gamma}/\langle \boldsymbol{a},\boldsymbol{a}\rangle)\boldsymbol{c} \sim \overline{K} = \overline{\overline{\boldsymbol{a}}} \cap \overline{\overline{\boldsymbol{b}}} \cap \overline{\overline{\boldsymbol{c}}}$$

in certain (explicit) circumstances for $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$, *so that (Fig. 4) the triangle sides* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$*; furthermore, the above* $\boldsymbol{a}'$, $\boldsymbol{a}''$, $\boldsymbol{b}'$, $\boldsymbol{b}''$, $\boldsymbol{c}'$, $\boldsymbol{c}''$ *are lines of two confocal line conics with the above* K *and* $K^* \sim \overline{K}$ *as foci.* □

But the further details would break up the framework of a didactical paper. We turn back to the topic in a research paper.

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A háromszög Kárteszi pontjairól, a három tükrözés tétele és geometriai algebra segítségével

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Kárteszi Ferenc (1907 – 1989) professzor úr, tanárom,
doktori témavezetőm emlékére.

Absztrakt. Idézzük fel az alábbi ismert feladatot: *Az ABC háromszög* (euklideszi $\mathbf{E}^2$) *síkjában szabályos háromszögeket rajzolunk kifelé az oldalakra, mondjuk* $AB\overline{C}$, $BC\overline{A}$, $CA\overline{B}$ *jelöléssel. Bizonyítsuk be, hogy az* $A\overline{A}$, $B\overline{B}$, $C\overline{C}$ *szakaszok egy K pontban metszik egymást. Ez a K pont az ABC un. izogonális pontja* (melyből az oldalak egyforma 120° szögben látszanak, hívják Fermat vagy Fermat-Torricelli pontnak is), *és melyre az* $AK + BK + CK$ *távolságösszeg a legkisebb*, *a sík bármely K-tól különböző pontját is tekintjük.*

Kárteszi professzor észrevette, hogy ez a K pont akkor is létrejön, ha szabályos háromszögek helyett egyenlő alapszögű egyenlő szárú háromszögeket rajzolunk ABC oldalaira kifelé, továbbá ez a Bolyai-Lobacsevszkij-féle hiperbolikus geometriában (sőt az $\mathbf{S}^2$ (gömbfelületi) szférikus síkon, lásd még erről Kálmán Attila [1] könyvét) is igaz. Ezért is javaslom a *Kárteszi pont* elnevezést. A magasságpont, súlypont speciális esetek lesznek. Rokonságban van a témával I. M. Yaglom (Jaglom) [5] Geometriai transzformációk II könyvének 83. szélsőérték feladata (módosított jelöléssel):

Egy adott ABC háromszög $(\mathbf{E}^2)$*síkjában keressük meg azt a K* pontot, *melyre az* $\alpha KA + \beta KB + \gamma KC$ *mennyiség a lehető legkisebb, ahol* α, β, γ *adott pozitív valós számok.*

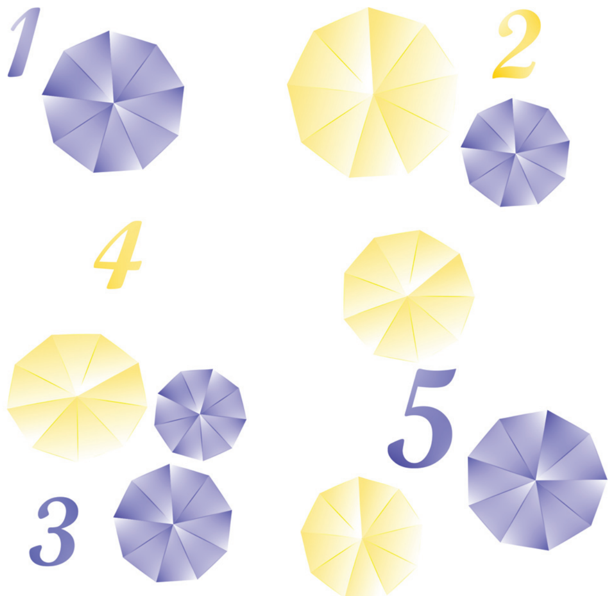
Ez a feladat általánosabb háromszög alakzatokhoz és analóg extremális K ponthoz vezet. Továbbá, és ez új eredményünk *a dolgozatban, a Bolyai János-féle "abszolút síkra"* (az $\mathbf{S}^2$ gömbre, $\mathbf{E}^2$, $\mathbf{H}^2$ síkokra, az $\mathbf{M}^2$ Minkowski és a $\mathbf{G}^2$ Galilei (vagy izotróp) síkokra egyaránt) *érvényes módon lehet megfogalmazni és megoldani a problémát a három tükrözés tétele* [3], [5] *és az un. geometriai (Grassmann-Clifford típusú) algebra segítségével* [4]. Így további nyitott kérdések

is felvetődnek. Ezzel követni kívánjuk Kárteszi Ferenc módszertani hitvallását (lásd pl. nagy nemzetközi sikerű [2] csodálatos könyvét):

> Induljunk ki egy természetes, elemi, szemléletesen is jól érthető feladatból. Ezután következhetnek a próbálkozások, matematikai kísérletek, eszközök és fogalmak kialakítása, majd a megoldás, esetleges elmélet, további alkalmazások és általánosítások,...

Kulcsszavak: háromszög geometria, három tükrözés tétele, geometriai (Grassmann-Clifford típusú) algebra, abszolút geometria, probléma-megoldás hasonlósági transzformációval

4.
Contemporary Challenges and Solutions in Mathematics Education



The *Traditional–Contemporary* Construct in Relation to the Learning Paradigms in the Discipline of Didactics of Mathematics

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Abstract. In Slovenia, both in the general pedagogical field and in the field of teaching and learning mathematics, a distinction was made between so-called *traditional* and *contemporary* teaching, which introduced various dichotomies into the pedagogical domain (e.g., passive and active learning, transfer of knowledge and independent learning, abstract and concrete mathematics teaching). The problematic nature of this phenomenon has been addressed in the past by some authors who considered such a distinction controversial. With the recognition that certain dichotomies are problematic, important steps have been taken to overcome them. This paper continues this process of overcoming by considering the *traditional–contemporary* construct in relation to some of the learning paradigms that have prevailed in the history of didactics of mathematics. After an initial definition of the *traditional–contemporary* construct and an outline of the emergence of the discipline of didactics of mathematics, we describe some of the most influential learning paradigms that have had a significant impact on the design and implementation of disciplinary research. We point out the problem of committing to a single learning paradigm and discuss the implications of adopting the *traditional–contemporary* construct. Finally, we argue for mathematical rigor and a scientifically grounded discussion of the direct and dialogic instruction of the teaching and learning of mathematics as a way of overcoming the ideological discourse of the *traditional–contemporary* construct.

Keywords: traditional–contemporary construct, learning paradigms, overcoming ideological discourse, teaching and learning mathematics

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1. Introduction

In recent years, attention has been drawn to the problematic nature of some ideas about teaching and learning, both in the general field of pedagogy and in the discipline of mathematics didactics[1] (e.g., Christodoulou, 2014; Egan, 2002; Hodnik & Krek, 2022). More recently, two authors have written about these ideas, Hodnik and Krek (2022), who have examined three established dichotomies in the field of didactics of mathematics: transfer of knowledge and independent discovery of knowledge, abstract and concrete mathematics instruction, and repetition in learning and learning as fun. The dichotomies assume that the first part of each pair represents a *traditional* and thus undesirable way of teaching and the second part represents a *contemporary*, desirable way of teaching. The *traditional school* (instruction, approach, model, pedagogy, etc.) is said to be based on transmission teaching that requires students to memorise "with little or even no understanding" (Štefanc, 2011, p. 116) and reproduce the facts learned (Štefanc, 2005, 2011). Teaching is assumed to be teacher-centred and based on explanations and the transmission of knowledge through one-way communication, in which students take a passive role (Krek & Hodnik, 2022; Štefanc, 2005). In contrast, the *contemporary school* is supposedly based on transformative teaching with a learner-centred approach, where teaching is based on learners' experiences and two-way communication, and learners are empowered to learn independently and actively (Kovač Šebart et al., 2004; Krek & Hodnik, 2022; Štefanc, 2005). The dichotomies highlighted by Hodnik and Krek (2022) and others (e.g., active and passive learning) can be captured by considering the social construct *traditional–contemporary*. This construct creates and reproduces scientifically unsupported value polarizations and is based[2] on a belief in progress that creates an ideological discourse (Kovač Šebart et al., 2004; Štefanc, 2005). By identifying and addressing the particular dichotomies that bring problematic ideas about teaching, important steps have been taken to overcome the discourse inherent in the thesis of the existence of a traditional and *contemporary* teaching. In this paper, we attempt to continue this process by considering the *traditional–contemporary* construct in relation to the learning paradigms that have emerged throughout history in the disciplinary field of didactics of mathematics.

2. The emergence of the discipline of didactics of mathematics

Although mathematic education has been present in human culture for more than two millennia, scientific research in this area is relatively new (Kilpatrick, 2020). The beginning of the development of a scientific discipline concerned with the study of the teaching and learning of mathematics – the discipline of didactics

[1] In Europe, the discipline known internationally as "mathematics education" has become known as "didactics of mathematics" (Niss, 1999).

[2] In a categorical and exclusionary manner, value polarizations characterise positively what is desirable within a particular paradigm and negatively what that paradigm should not retain (Kovač Šebart et al., 2004; Štefanc, 2005).

of mathematics – is dated to the turn of the 20th century. At that time, in response to the need for better qualified mathematics teachers, the first mathematics didactics lectures were given at universities around the world (Kilpatrick, 1992; Schubring, 1983). Kilpatrick (1992) cites this as one of the main reasons for the beginning of disciplinary research, because even then university professors were committed to both teaching and research. Thus, at the turn of the millennium, the first disciplinary research began, and mathematics education became established as a scientific discipline. The first researchers in the field were mathematicians and psychologists (Kilpatrick, 1992; Schubring, 1983), which had an important impact on research design and on the understanding of what counts as scholarly work (Kilpatrick, 1992).

In the 20th century, mathematics education has been an important area for psychological research, in which various psychological theories have been applied to and tested in the area of mathematics teaching and learning (De Corte et al., 1996; Kilpatrick, 1992). In what follows, we briefly review some of the most influential psychological learning paradigms to illustrate the diversity and richness of theoretical assumptions about learning. We want to caution readers that our goal is not to provide a systematic account of all learning paradigms and their foundations for explaining learning – if such a goal is even attainable today – but rather to discuss some of the most influential theoretical frameworks that have had a significant impact on the design and conduct of disciplinary research. Based on this overview, we problematize the existence and reproduction of the *traditional–contemporary* construct. We believe that this, in addition to identifying specific problematic conceptions of teaching, as Hodnik and Krek (2022) have done, represents an important milestone in overcoming the existing discourse on *traditional* and *contemporary* teaching.

3. Learning paradigms

In the relatively short history of the discipline of didactics of mathematics, several paradigms have influenced its development and research. First, the **behaviourist theoretical paradigm** (Kilpatrick, 2020; Schubring, 1983) had a significant influence and was the dominant paradigm in psychology from the early 20th century to the late 1950s, with representatives such as Pavlov, Watson, and Skinner (De Corte et al., 1996; Schuh & Barab, 2008). Although this dominance was most characteristic of the United States, the paradigm was also well known in Europe (De Corte, 1996). Behaviourists considered the behaviour of individuals as the object of study, and because their research was to be based on objectively measurable characteristics of individuals (De Corte et al., 1996; Ertmer & Newby, 1993; Schuh & Barab, 2008), emotions, thinking, and other mental processes were deliberately excluded from the studies (De Corte et al., 1996). As a result, we can understand the behaviourist understanding and explanation of learning as “changes in either the form or frequency of observable performance” (Ertmer & Newby, 1993, p. 55) or “as having an organized collections of connections among elementary mental or behavioural units” (Greeno et al., 1996, p. 17). Learning is viewed in terms of the stimulus, the response to the stimulus, and their association, and much attention has been devoted to explaining how this association is made, reinforced, and maintained (Ertmer & Newby, 1993). Greeno et al. (1996) argue that behaviourist research has made an important contribution by emphasising the importance of teacher feedback and learning in small steps (sequential learning) and by emphasising that learning can also occur unintentionally (incidental learning). Mayer (1992) notes that although the influence of the behaviourist paradigm declined significantly in the second half of the 20th century, remnants of this paradigm, such as the emphasis on the automation of basic skills (e.g., in reading, writing, and arithmetic), are still part of contemporary theories of teaching and learning. Mayer (1992, 1996), who examined the various concepts of learning in terms of the metaphors that best explain learning, identified in behaviourism the metaphor of *learning as response strengthening*.

In the late 1950s, the so-called cognitive revolution in psychology is said to have taken place (Mayer, 1996; Schuh & Barab, 2008), and in the 1960s and 1970s, the dominant position of behaviourism was replaced by **information processing theory** (De Corte, 1996; Mayer, 1992, 1996; Schuh & Barab, 2008), which some authors equate with cognitivism (e.g., Schuh & Barab, 2008). This theory, unlike behaviourism, has also had a strong influence in Europe (De Corte, 1996). It conceptualises learning in a different way than behaviourism; Mayer (1996) describes it through the metaphor of *learning as knowledge acquisition*. Information processing theory explains learning in the context of a technology that was new for the time – the computer (De Corte et al., 1996). The focus of the study was no longer on individual behaviour and the stimuli that trigger it, but on the mind, which, like a computer, processes the acquired information and stores it in memory (De Corte, 1996; Mayer, 1996; Schuh & Barab, 2008). Researchers have attempted to explain the process of knowledge acquisition, transformation, organisation, encoding, and retrieval (Ertmer & Newby, 1993; Schuh & Barab, 2008), focusing on features not previously examined within the behaviourist theoretical paradigm (De Corte, 1996; Schuh & Barab, 2008). In addition, learners' beliefs, attitudes, and values were also examined because they were thought to have a significant impact on learning (Winne, 1985 in Ertmer & Newby, 1993). The goal of instruction in information processing theory is understood to be similar to that of the behaviourist paradigm – to transfer knowledge to learners as effectively as possible (Bednar et al., 1991). Theorists in both paradigms believe this is accomplished by analysing, breaking down, and simplifying content into its basic building blocks, where "behaviourists would focus on the design of the environment to optimise that transfer, while cognitivists would stress efficient processing strategies" (Ertmer & Newby, 1993, p. 1). Thompson et al. (1992) state that the emphasis on feedback is the most obvious commonality between the two paradigms, with behaviourists emphasising its importance in changing behaviour and cognitivists emphasising its importance in supporting thinking processes.

In the 1970s and 1980s, the emergence of a new understanding of learning, namely *learning as knowledge construction* (Mayer, 1992, 1996), led to heated epistemological debates between cognitivists and radical constructivists (Cobb, 2007; Ernest, 2018). In contrast to behaviourism and cognitivism, which are based on objectivism and assume the existence of a real world outside the individual (Schuh & Barab, 2008), **radical constructivism** takes an agnostic stance toward reality (Riegler, 2005), neither denying nor accepting its existence (von Glasersfeld, 1991). In the non-acceptance of objectivism lies the so-called radical position of constructivism, which in this respect differs from so-called trivial constructivism (also cognitive constructivism, see (Taber, 2006)), which assumes that knowledge is constructed but maintains the assumption that one's "conceptual constructions can or should in some way represent an independent, 'objective' reality" (von Glasersfeld, 1991, p. 5). One of the most well-known founders and proponents of radical constructivism, who also related his theory to mathematics education, Ernst von Glasersfeld, views knowledge as a means of organising one's experiences rather than a reflection or representation of reality that is independent of one's experiences (von Glasersfeld, 1994). Therefore, the author also argues that "knowledge does not reflect an "objective" ontological reality" (von Glasersfeld, 1991, p. 5), since knowledge is not "a commodity that is found ready-made but must be the result of a cognizing subject's construction" (von Glasersfeld, 1986, pp. 109). In other words, radical constructivism assumes that people create meaning rather than acquire it (Ertmer & Newby, 1993). Constructivism also assumes that the construction of knowledge depends on previous experiences and on the existing mental structures and beliefs of the individual. Unlike behaviourists and cognitivists, constructivists do not argue that mental processes can create meanings that are external to the individual (Jonassen, 1991), but that meaning is something created by the learning subject (Ertmer & Newby, 1993). Thus, knowledge is not understood as a reflection of reality, as something that matches it, but as something that fits reality (von Glasersfeld, 1991). In the 1980s, therefore, research on the teaching and learning of mathematics took on new dimensions, with mathematical thinking becoming the central object of study and new research methods, especially clinical interviews, coming to the fore (Kilpatrick, 2020; Schoenfeld, 2016; Schubring, 1983). By adopting a cognitive perspective and constructivist foundations, researchers in the teaching and learning of mathematics were encouraged to examine metacognition, belief systems, and problem solving (Schoenfeld, 2016). As Inglis and Foster (2018) demonstrated in their empirical study based on the analysis of nearly 4 000 documents, constructivism has become the dominant paradigm in the field of mathematics education research. They found that research on mathematics teaching and learning in the context of this paradigm peaked in the period from the late 1980s to the mid-1990s, after which interest in research in this area declined.

As with earlier paradigms, the constructivist understanding of learning has met with certain concerns and criticisms, such as how the individual knowledge constructions that learners arrive at lead to common[3] mathematical concepts or procedures (Putnam et al., 1990 in De Corte et al., 1996). The latter concern has been addressed by explaining learning as a social process, emphasising the importance of social interaction, negotiation, and collaboration, and viewing learners as members of a mathematical community and culture (De Corte et al., 1996). A number of theories were developed, such as **social constructivism**, **sociocultural**

[3] Radical constructivists would argue that we cannot check whether students have really mastered the same mathematical concepts or procedures.

theory, **situated theory**, and **distributed cognition**, all emphasising in their own way the importance of social, cultural, historical, physical, and/or symbolic context in learning (Cobb, 2007; De Corte et al., 1996). As a result, at the beginning of the millennium, mathematics education began to take a more holistic view of the factors that influence students' mathematical achievement (Schoenfeld, 2016), and researchers focused their attention on examining the impact of social and cultural factors, what Lerman (2000) referred to as the social (or Gutiérrez (2013) as sociopolitical) turn in mathematics education research. The presence of this turn was confirmed by the research of Inglis and Foster (2018) – they noted that after a decline in interest in research on mathematics teaching and learning in relation to constructivist theory, interest in research on mathematics teaching and learning in the context of a sociocultural paradigm has increased significantly.

4. Acceptance of multiparadigmatism

A brief review of some paradigms shows that each paradigm explains learning in different ways and from different perspectives: 1) the stimulus, the response to the stimulus, and their association, 2) the reception, transformation, encoding, and organization of information, 3) the construction of knowledge that "fits" reality, 4) social, cultural, historical, and other factors; or through the metaphors identified by Mayer (1992, 1996): *learning as response strengthening*, *learning as knowledge acquisition*, *learning as knowledge construction*, and *learning as social negotiation*. Kirshner (2015) points out the problem of the multiparadigmatic nature of learning theory in psychology. Following Kuhn's (1970) theory, he argues that psychology is a pre-paradigmatic science despite opposition from some psychologists because incommensurable paradigms have been established that explain learning in different ways, but none of them is so holistic that its establishment would cause the demise of the others (Kirshner, 2015). Kirshner (2015) asserts that attempting to establish only one of the existing paradigms in education leads to conflict and confusion as a number of teaching models are developed based on different learning paradigms that prove to be contradictory and provide conflicting advice for teaching. Cobb (2007) sees the problem not in the theoretical background of the various perspectives, but in the assumed connection these perspectives are supposed to have with classroom practice. He refers to the process of transforming descriptive perspectives on learning into direct prescriptions for teaching as category error and, like Ertmer and Newby (1993), points out that different paradigms sometimes provide similar prescriptions for pedagogical practice, which Cobb (2007) argues leads to undefined pedagogies based on ideologies rather than empirical evidence.

Different learning theories differ in their philosophical assumptions (Ernest, 2018), which are based on different beliefs about the nature of the world, the individual, and their relationship to each other (Schuh & Barab, 2008). The relevance of the beliefs held by a particular theory cannot be demonstrated by rational argument or empirical research, so it cannot be proven that one theory is better than another (Kirshner, 2015; Schuh & Barab, 2008). The findings we arrive at in the discipline of didactics of mathematics, such as various theories and instructional models, are not theorems, they do not have the property of generality (Niss, 1999), and thus it is not useful to ask which theory is best, but which theory is most effective given the specific learners to whom we are teaching a particular content (Ertmer & Newby, 1993). No two learners have the same needs (Sfard, 1998), and the teaching and learning of mathematics is always situated in a specific context and dependent on a number of factors, which also limits the generalizability of disciplinary findings (Niss, 1999). Kirshner (2015) suggests accepting the fact that there are several different theories of learning and not just advocating one theory, but considering all theories as legitimate. The author points out that today we take for granted that we are describing learning as a complex whole, and points out that this was not the case in the past. In the past, individual paradigms have attempted to theorise learning independently (Kirshner, 2015) and have established themselves as perspectives that transcend the boundaries of their predecessors (Cobb, 2007). We believe it would be beneficial to conceptualise different paradigms as theoretical frameworks that guide, organise, and structure the researcher's or teacher's view while blurring the rest (Spangler & Williams, 2019), leading to the viewpoint that paradigms only partially address mathematics teaching and learning. Sfard (1998) and Kirshner (2015) suggest abandoning the hope that a single theoretical framework or understanding of learning will guide the design and delivery of mathematics education and abandoning efforts to define a single theoretical position. Lester (2005) and Cobb (2007) suggest that rather than adopting a single theoretical perspective, we should behave more like a bricoleur – a handyman who solves problems with the tools at his disposal. Theoretical perspectives should be conceptual tools, and the mathematics teacher should adapt and apply ideas from a variety of sources depending on the interests and goals of his or her teaching (Cobb, 2007).

The above provides an important starting point for understanding and addressing the *traditional–contemporary* construct. The construct encompasses various dichotomies aimed at orienting teachers to so-called modern, desirable, or progressive forms of teaching, and these orientations are often found in discussions of particular learning theories. Hodnik and Krek (2022) point out that some of the dichotomies are placed in the context of an overly simplistic understanding of constructivist theory, and Hattie (2009) reports that every year he encounters prospective teachers who have adopted the mantra that direct instruction is a bad way to teach and constructivist instruction is a good way to teach. We ourselves find that the behaviourist paradigm, with its direct teaching of facts, is a frequent reference point and sometimes even synonymous with *traditional* and undesirable teaching. Discussions that suggest the superiority of one understanding of learning (e.g., constructivist) over another (e.g., behaviourist) lack any rational or empirical basis (Kirshner, 2015; Schuh & Barab, 2008) and are therefore unscientific. The latter conclusion is of explicit relevance to the discretization of the *traditional–contemporary* construct which, we note, derives an important part of its essence precisely from the value opposition of different understandings of learning.

4.1. Direct instruction and teaching facts

In the following, a concrete example from the field of teaching and learning mathematics will be used to show how the ideological discourse of value polarization, in which direct instruction and the teaching facts have negative connotations, can be overcome.

Many of the criticisms against teaching facts are addressed in Christodoulou's (2014) book when she discusses the beliefs that learning facts prevents understanding, that the teacher should not take the lead in teaching facts because it forces students to be passive, and that memorization is not as important today because information is more readily available than in the past. The author responds to these beliefs with a critique, pointing out that learning facts is critical to conceptual understanding and subsequent problem solving, and that students' memory is overloaded when they learn facts independently, making learning less effective than with teacher support (Christodoulou, 2014). A similar issue is raised by Hattie et al. (2017) when they point out that there are divided opinions among researchers about how teachers should approach instruction – with explicit instruction or starting from a mathematical problem. The authors point out that such black-and-white debates are not appropriate, as learning goals and students' background knowledge must be considered when choosing an instructional approach (Hattie et al., 2017). Citing the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), the authors advocate for *mathematical rigor* – "a balance among conceptual understanding, procedural skills and fluency, and application with equal intensity" (Hattie et al., 2017, p. 3). Furthermore, Hattie et al. (2017) point out that in the field of mathematics teaching and learning, there is a dichotomous debate about "direct instruction versus dialogic instruction" with researchers and teachers arguing for one position or the other. Hattie (2009) points out that direct instruction is mistakenly viewed as frontal teacher talk, and in Hattie et al. (2017, p. 3), he argues that "direct instruction includes much more than showing and telling students how to perform the computational skills they are learning" adding that there is an important distinction between teaching and telling. He defines direct instruction as instruction in which "the teacher decides the learning intentions and success criteria, makes them transparent to the students, demonstrates them by modeling, evaluates if they understand what they have been told by checking for understanding, and re-tells them what they have been told by tying it all together with closure" (Hattie et al., 2017, p. 24). Although direct and dialogic instruction differ in some aspects, both have a similar goal – student mastery of mathematics (Hattie et al., 2017). Based on a synthesis of over 800 meta-analyzes conducted by Hattie (2009), Hattie et al. (2017) conclude that the effect sizes of both instructions are above the 0.40 level (an important point where the so-called Zone of Desired Effects begins), suggesting that both instructional models have been shown to be significant in classroom practice. The authors advocate for thoughtful integration of both instructional models in the classroom, as both can contribute to the learning process in their own way (Hattie et al., 2017). Hattie et al.'s (2017) scientifically informed discussion does not postulate the various instructional models as opposing alternatives or assign them positive or negative value. In this and in their plea for mathematical rigor and discussion of direct and dialogic instruction, we see an example of how the ideological discourse of value polarization can be overcome in mathematics education.

5. Conclusion

The aim of this paper was to show that throughout the development of the discipline of didactics of mathematics there have been different understandings and explanations of learning and, consequently, different guidance for teachers in planning and delivering their lessons. Different learning theories are based on different ontological, epistemological, and other philosophical assumptions that cannot be confirmed or refuted by rational argumentation or empirical research, so the search for the best theory is not productive (Kirshner, 2015; Schuh & Barab, 2008). Like Kirshner (2015) and Cobb (2007), we argue that mathematics educators should accept that different learning paradigms exist and understand them as conceptual tools that they can use depending on the needs of their own teaching. The juxtaposition of concepts, teaching methods, ways of working, and teaching approaches (e.g., direct teaching of facts versus independent discovery of knowledge) that are presented as mutually exclusive alternatives, at least implicitly stating that one of the alternatives is desirable and modern and the other is not, cannot contribute to the formation of teachers who are able to thoughtfully and successfully plan and deliver their lessons according to the needs of their students and the content to be taught. By accepting and using the *traditional–contemporary* construct, which places different concepts in the context of polarising values, we are accepting and reproducing an ideological discourse and thus distancing ourselves from quality teaching based on scientifically grounded truths. Like Spangler and Williams (2019), we argue for viewing paradigms as theoretical frameworks that direct our view to certain aspects of learning while blurring out others, forcing us to accept that each paradigm is inherently limited to only certain aspects of human learning. In this latter position, we see a way out of the current tendency to establish, defend, and glorify only one paradigm, pedagogy, instructional model, or teaching method.

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Konstrukt *tradicionalno–sodobno* v odnosu do paradigem učenja na področju discipline didaktika matematike

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Povzetek. V Sloveniji se je na splošnem pedagoškem področju, kot tudi na področju poučevanja in učenja matematike, uveljavilo razlikovanje med t. i. *tradicionalnim* in *sodobnim* poučevanjem, ki je v pedagoški prostor vneslo različne dihotomije (npr. pasivno in aktivno učenje, prenos znanja in samostojno učenje, abstraktno in konkretno poučevanje matematike). Problematiko obstoječega fenomena so v preteklosti že obravnavali nekateri avtorji in nakazali, da je takšno razlikovanje sporno. S prepoznavo problematičnosti posameznih dihotomij, so bili narejeni pomembni koraki k njihovi premostitvi. S pričujočim prispevkom želimo z obravnavo konstrukta *tradicionalno–sodobno* v navezavi na nekatere paradigme učenja, ki so se skozi zgodovino uveljavile na področju discipline didaktike matematike, proces preseganja nadaljevati. Po začetni opredelitvi konstrukta *tradicionalno–sodobno* in orisu nastanka discipline didaktike matematike, opišemo nekatere najvplivnejše paradigme učenja, ki so pomembno vplivale na snovanje in izvedbo disciplinarnih raziskav. Izpostavimo problematičnost prizadevanj po uveljavitvi zgolj ene izmed njih in razpravljamo o posledicah sprejemanja konstrukta *tradicionalno–sodobno*. Na koncu se zavzemamo za matematično strogost in znanstveno utemeljeno razpravo o direktnem in dialoškem poučevanju in učenju matematike kot način za preseganje ideološkega diskurza konstrukta *tradicionalno–sodobno*.

Ključne besede: konstrukt *tradicionalno–sodobno*, paradigme učenja, preseganje ideološkega diskurza, poučevanje in učenje matematike

Decision Trees in Research on Mathematics in Primary School

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Abstract. Mathematics is a compulsory subject and as such runs through the entire vertical of formal education, indicating its importance both in daily life and in its contribution to the accelerated development of modern society. Numeracy has not only played a great role in the past, but is also of great importance for the future, as it is necessary to understand the environment in which we live and is one of the foundations for the development of life skills of each individual. Educational data mining is a research area in which data mining methods are applied to educational data. Although the use of data mining in education has increased in recent years, the potential of these methods is not yet fully realized, especially in studies with primary school pupils. Decision trees, regression trees, and classification trees are methods that are applicable in this research area. The advantages of these methods are a very clear visual overview of the results for the description of the model, making them easy to analyze and interpret, and they can be very easily combined with other research methods. Since there is a lack of research that incorporates the aforementioned methods in mathematics research, especially in primary education, the purpose of this paper is to analyze the use of decision trees, regression trees, and classification trees in primary school mathematics research. A search of five academic databases (WoS, Scopus, Emerald, Eric and SpringerLink) found only 11 papers that meet the criteria. This review covers their analysis and answers research questions about the frequency of use of these methods and the purpose of their use.

Keywords: classification tree, decision tree, data mining, primary school mathematics, regression tree

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1. Introduction

The status of a compulsory subject and its presence throughout the educational vertical expresses the importance of mathematics for everyday life and functioning. The curriculum of the subject Mathematics (Ministry of Science and Education of the Republic of Croatia, 2019) highlights its important role in the progress of society, both in the past and in the present and the future. In addition, mathematical knowledge is necessary to understand the world that surrounds us, and learning mathematics develops mathematical knowledge and skills that are important for personal, social, and professional life (Ministry of Science and Education of the Republic of Croatia, 2019). Mathematical literacy is essential for life skills, mathematical strategy, lifelong learning, openness to new technologies, and potential realization, according to the curriculum. Learning and teaching mathematics fosters creativity, accuracy, systematicity, abstract and critical thinking, which helps solve daily and social problems. Despite the importance of mathematics and mathematical literacy, many students have problems in adopting mathematical concepts and have difficulties in learning mathematics. Teachers also have difficulty building a teaching process that will make it easier for students to learn math. In recent decades, technological innovations have helped teachers and students improve teaching and learning in mathematics and science, transforming learning from a teacher-centred to a learner-centred approach. Learner-oriented approach is efficient, smart, improves performance, and boosts pleasure. It allows learning-on-the-fly and leverages technology to save time and money (Sweta, 2021).

Data mining extracts hidden, unknown, and likely important information from enormous amounts of data. It uncovers hidden trends and patterns in business, medicine, and other fields using a massive knowledge base, advanced analytical skills, and domain knowledge (Kamath & Kamat, 2016). Educational Data Mining (EDM), the fastest growing research topic in computer science, uses data mining methods (DM) to collect, explore, and use data for educational purposes. It contains many innovative ideas for teaching, such as how to assess learners' needs, how to teach properly, and what learning approaches to use for good lesson planning and delivery (Sweta, 2021). Developing methods for exploring unique educational data and using them to improve educational experiences holds great potential for education (Kamath & Kamat, 2016). Aleem & Gore (2020) describe prediction methods as methods for development a model for the attribute to be predicted based on the dataset's other properties. Prediction algorithms are most often employed to address student dropout and academic performance. Training is done on the dataset with known values of the attribute to be predicted. The initial model predicts attribute values for another dataset that may not have known values. Prediction methods can be used for classification (naïve Bayes, fuzzy rule learning classifiers, rule induction, decision rules, decision trees, random forest, step regression, logistic regression, statistical, neural networks...), regression (linear regression, and regression trees), density estimation (vector quantization, Parzen windows, density-based clustering methods...) and latent knowledge estimation (performance factors assessment, Bayes net, Naïve Bayes, Bayesian Knowledge Tracing...).

2. Literature review

Hernández-Blanco et al. (2019) reviewed researches on deep learning (DL) in EDM from its inception to 2018 and concluded that it has only been used to solve four types of problems: predicting student performance, detecting poor behavior, making suggestions, and automating assessment. Their study also revised the major EDM databases. As in other research domains, some of them are publicly available for the scientific community to reproduce experiments, while others were produced ad hoc for specific investigations. In EDM, sensitive information about (underage) pupils makes datasets difficult to share. Their work also provides an introduction to DL approaches, an examination of the DL architectures used in each task, a discussion of the most frequent hyperparameter configurations, and a list of frameworks to help construct DL models.

Shin and Shim (2021) reviewed mathematics and science instructional data mining literature. Data mining in mathematics and science education has been used to identify students' behavior and thinking, identify success criteria and automatically grade student writing. Recently, researchers have employed text mining to build learning programs for students as well as educators. Researchers used classification, text mining, and clustering for data mining. Science education had more data mining studies than mathematics education.

Using a general system theory framework, Xu and Ouyang (2022) conducted a systematic analysis of AI-STEM research between 2011 and 2021 to comprehend artificial intelligence (AI) applications in STEM education. They have noted a decade-long increase in the use of AI (e.g., educational robotics, intelligent tutoring systems (ITS), student performance recognition, and others) in STEM education. It was noted that automation was only utilized for lecture-based training, whereas educational robots were predominantly utilized for problem-based learning.

Khan and Ghosh (2021) reviewed EDM research on assessing and predicting student performance in their article. They focus mainly on instructional education research and the literature on predicting learner performance after a course has begun appears to be extensive. However, early prediction before the course begins still remains a challenge.

The objective of Bin Roslan and Chen's (2022) systematic literature review was to identify the recent trend in research, the most studied factors, and the methods used to predict student academic success between 2015 and 2021. The results indicate that the focus of current studies is on identifying characteristics that affect student performance, the performance of data mining algorithms, and data mining in relation to e-learning systems. In addition, it is found that student academic dossiers and demographic information are the most influential factors on student performance. Classification is the most commonly used data mining technique, and decision tree is the most commonly used data mining method according to their research.

The use of data mining in education has increased, but its potential, especially in primary school subjects, has yet to be realized. Most literature reviews on data mining in education have focused on its general application, and very few reviews examine the application of specific EDM methods in primary school mathematics. Methods such as decision trees are applicable in this research area, and there is a gap in the literature that explores the application of these methods in primary school mathematics education. The purpose of this paper is to examine the application of decision trees in research dealing with primary school mathematics and to answer the research question:

RQ1. What is the distribution of decision trees in primary school mathematics research by year of publication?

RQ2. What is the purpose of using decision trees in primary school mathematics research?

3. Methodology

This review was conducted in January 2023, and followed the methodology outlined in the *Preferred reporting items for systematic review and Meta-Analysis (PRISMA)* to find and summarize relevant articles (Moher et al., 2015). Five academic databases were used for the literature review: Emerald, ERIC, Web of Science (WoS), Scopus, and SpringerLink. The article search was not limited by the year of publication of the articles. The search was initiated with the following keywords: decision tree, regression tree, classification tree, mathematics, primary school and elementary school. Academic databases presented the following number of papers: Emerald 34, ERIC 2, WoS 142, Scopus 27 and SpringerLink 1912. Inclusion and exclusion criteria were utilized to choose relevant studies (the criteria are detailed in Table 1).

Table 1. Inclusion and exclusion criteria.

Inclusion criteria	Exclusion criteria
• open access research papers • decision trees, regression trees or classification trees used in research • primary/elementary school research • mathematics research • empirical research • English-written research papers	• review, theoretical, and conceptual papers • research papers without an educational focus on learning or teaching mathematics • research papers without full text access • research papers not written in English language

Based on the inclusion/exclusion criteria, the authors decided whether an article should be included in the review. A total of 11 articles were considered for this review. The selection of articles for this review is shown in Figure 1.

In the final step, the articles were classified according to the purpose of the research.

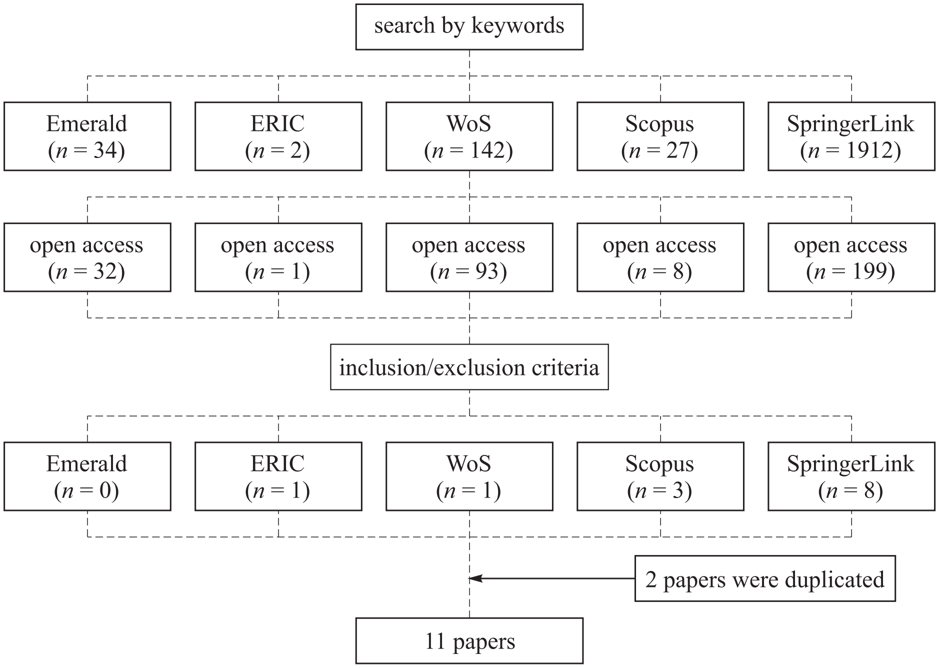

Figure 1. Paper selection processes.

4. Results and discussion

A total of 2117 results were retrieved from five scientific databases (Emerald, ERIC, WoS, Scopus, and SpringerLink). For this review, 11 articles were considered based on inclusion and exclusion criteria. Table 2 provides a complete overview of the articles included in the analysis.

Table 2. Papers included in the analysis.

Year	Author(s)
2013	Käser et al.
2016	Peeters, Degrande, Ebersbach, Verschaffel, & Luwel
2021	Bignardi, Dalmaijer, Anwyl-Irvine, & Astle
2021	Ko, Choi, & Kaji
2020	Haridas, Ramaraju, & Nedungadi
2020	Torres-Ramos et al.
2022	Aßmann, Gaasch, & Stingl
2022	Cukurova, Khan-Galaria, Millán, & Luckin
2022	Gnambs & Lockl
2022	Najm et al.
2022	Yuhana, Oktavia, Fatichah, & Purwarianti

RQ1. What is the distribution of decision trees in primary school mathematics research by year of publication?

Figure 2 shows the distribution of contributions by year of publication. The data show that the frequency of using decision tree methods in primary school mathematics research has increased in recent years. However, the number of studies using this method is minimal, indicating an underutilization of its potential in educational research, especially research related to mathematics education.

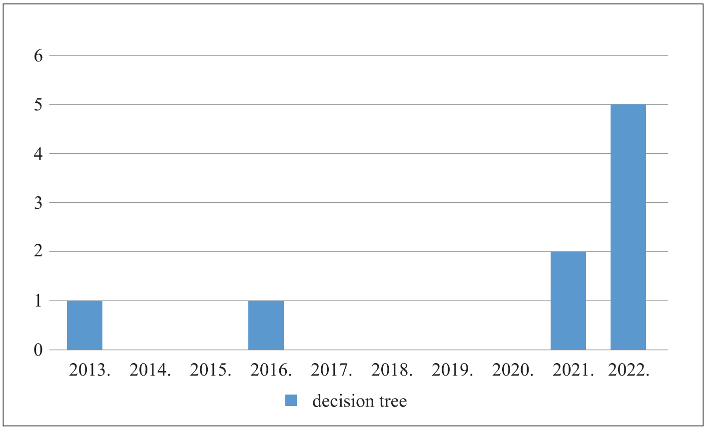

Figure 2. Distribution of papers by year of publication.

RQ2. What is the purpose of using decision trees in primary school mathematics research?

Analysis of the papers shows that decision trees are used for both prediction and classification.

Käser et al. (2013) developed a control algorithm and student model to improve children's mathematical education and help them learn more effectively. A computer-based numerical cognition training system for children with developmental dyscalculia or mathematical difficulties integrates the adaptive system. A dynamic Bayesian network with domain knowledge drives an automatic online control system in the student model. Based on the estimated knowledge, the system selects tasks and exercises. The student's behavior is monitored to detect predictive control patterns. Adaptive and personalized training increases success and motivation. Extensive testing of input data verifies the quality of results and the benefits of optimized training.

Haridas et al. (2020) conducted a three-year longitudinal study of 2,123 Indian students using the AmritaITS intelligent tutoring system. The study used AmritaITS interaction logs to (1) predict student performance in math and English through summative and formative assessments, (2) predict students who might fail the final test, and (3) screen students for reading problems.

The recently developed term *supoza* refers to students who have given up on learning mathematics, despite the lack of a clear definition for the term (Ko et al., 2021). A statistical analysis of the similarities among students who have given up on learning mathematics was undertaken in study conducted by Ko et al. (2021). These authors aimed to identify differences between two groups' affective domains: individuals who admitted to giving up on learning mathematics and those who maintained. Based on results, they found that the emotional domain for mathematics learning can be used to categorize children who have given up on learning mathematics. They discovered that a statistical model based on affective domain questions can predict whether a student will give up on learning math. This suggests that affective factors alone can explain *supoza* and emphasizes the importance of the affective domain of math learning.

In their article, Bignardi et al. (2021) evaluate the results of a tablet app developed for the Resilience in Education and Development project (RED). The RED app contains 12 cognitive tasks for 7-13-year-olds to complete independently during a 1-hour school lesson. Despite the lack of participant engagement, the data were of very good quality. Internal consistency metrics showed moderate or high reliability for most tablet results. The cognitive skills tested with the tablets were able to explain more than 50 % of the variance in academic achievement reported by teachers. They concluded that tablet-based group cognitive assessments of children are a low-cost, reliable, and valid technique for obtaining large data sets for modern psychology.

The study by Najm et al. (2022) used Multidimensional Online Analytical Processing (MOLAP) and data mining methods, decision trees, to predict student achievement and grades using educational data mining. The study compared data mining methods to find the best method for predicting student performance.

The study by Peeters et al. (2016) examined benchmark-based number line estimation (NLE) strategies for second graders. Their study is one of the first to show that children's benchmark-based estimation techniques using natural numbers in the NLE affected their performance and that a decision tree was used for the classification strategies.

Torres-Ramos et al. (2020) introduced a different characterization of the EEG signal based on a graphical measurement system and a decision tree classification based on these features. Their proposal sought to analyze the differences between groups in terms of brain connection networks related to mathematical competencies in elementary school students.

In their study, Yuhana et al. (2022) developed a tiered system for reviewing student responses. The primary contribution is the comparison of trees from two multimodal inputs and the automatic evaluation of student responses. It collected operands from mathematical stories and then categorized the operator using Random Forest to create a key and convert it into a tree. The OCR library extracts the text from the learner's answer image and determines the operand, operator, and result to create the answer tree. The trees are matched for automatic scoring. In this study, 500 questions, 300 training data, and 200 test data were used. Questions with an operator have five classes: subtraction, addition, division, multiplication, and mixed. The study found a categorization accuracy for mixed operators of 68.8 % and for simple operators of 84.31 %. Due to limitations in image processing of student responses, tree matching accuracy was 78.12 %.

Cukurova et al. (2022) analyzed teacher behavior and categorized effective and ineffective tutoring sessions. They used sequential behavior using the CM-SPAM pattern mining algorithm to perform an analysis of tutoring sessions and categorized tutoring sessions as effective or ineffective by classifying them using the J-48 and JRIP decision trees.

In the study by Aßmann et al. (2022) a data-expanded Bayesian estimation method for missing values in multilevel latent regression models was proposed. The testing scheme is extended to include samples from conditional distributions of missing values, and nonparametric classification and regression trees should model the full conditional distributions of missing values. This makes the latent quantity information a sufficient statistic. In terms of statistical efficiency and computation time, this Bayesian approach outperforms both full case analysis and multiple imputation methods. Mathematical competency data support the proposed approach.

In their study, Gnambs and Lockl (2022) examined the lagged effects of reading and mathematics in grades 5-12 among German students on the National Educational Panel. The bidirectional effects were negligible to small, depending on the causal estimator. Mathematicians who scored higher on one measure had slightly better reading scores at the subsequent measurement time point. In comparison, a within-person analysis showed that both abilities were able to predict increases in each other's longitudinal performance in earlier grades, but the results were reversed in later grades. The results suggest that transfer effects at the secondary level between reading and mathematics are minimal and model-dependent.

In education, predictive studies are routinely used to provide strategic information about teaching and learning processes and what characteristics predict educational outcomes such as academic achievement or school dropout. Predictive studies in education typically employ a range of methods similar to those used in other fields. Although the decision tree method is a widely used standard method in data mining and has been used in data science since the 1980s, it is not a traditional tool for predictive research in education (Gomes & Almeida, 2017). This review of the literature supports these claims. The results show that very few studies have used decision trees for the purpose of prediction.

Many students struggle to understand mathematics, and teachers struggle to design mathematical learning processes. Technology in education has improved mathematics and science education in recent decades, and data mining contains creative ideas for teaching, such as how to analyze learners' needs, teach them well, provide them with learning materials, and use learning methods for effective teaching. Unlike traditional methods, the use of decision trees does not require assumptions about the data or an a priori target variable prediction model. Instead, the data is used to create a model that is suitable for dealing with non-linear correlations between variables and provides intuitive, easy-to-understand results. The trees are transparent and do not require statistical understanding, making them suitable for communicating findings to a wide range of individuals, including education managers, teachers, parents, or students (Gomes & Almeida, 2017). As mentioned earlier, the decision tree method offers numerous advantages for research in education. Unfortunately, the results indicate an underutilization of this method.

5. Conclusion

The purpose of this paper was to investigate the use of decision trees in primary school mathematics research. After searching five academic databases (WoS, Scopus, Emerald, Eric, and SpringerLink), only eleven publications met this review research criteria. In recent years, decision trees have been increasingly used in elementary school mathematics research. But despite their advantages, they are still not a common technique for predictive research in education.

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Stabla odlučivanja u istraživanjima matematike u osnovnoj školi

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Sažetak. Matematika je obvezni nastavni predmet i proteže se kroz cijelu vertikalu formalnog obrazovanja. To ukazuje na njezinu važnost u svakodnevnom životu, ali i na njezin doprinos ubrzanom razvoju suvremenog društva. Matematička pismenost ne samo da je imala veliku ulogu u prošlosti, već je i od velike važnosti za budućnost. Potrebno je razumjeti okruženje u kojem živimo, a matematička pismenost jedan je od temelja za razvoj životnih vještina svakog pojedinca. Rudarenje obrazovnih podataka je istraživačko područje u kojem se metode rudarenja podataka primjenjuju na obrazovne podatke. Iako se korištenje rudarenja podataka u obrazovanju posljednjih godina povećalo, potencijal ovih metoda još nije u potpunosti ostvaren, posebno u istraživanjima koja se odnose na učenike osnovnih škola. Stabla odlučivanja, regresijska i klasifikacijska stabla metode su koje su primjenjive u ovom istraživačkom području. Prednosti ovih metoda su vrlo jasan vizualni pregled rezultata za opis modela što ih čini jednostavnim za analizu i tumačenje, a vrlo lako se mogu kombinirati s drugim istraživačkim metodama. Budući da nedostaje istraživanja koja uključuju gore navedene metode u istraživanje matematike, posebno u osnovnom obrazovanju, svrha ovog rada je analizirati korištenje stabala odlučivanja, regresijskih i klasifikacijskih stabala u istraživanjima matematike u osnovnoj školi. Pretragom pet akademskih baza podataka (WoS, Scopus, Emerald, Eric i SpringerLink) pronađeno je samo 11 radova koji zadovoljavaju kriterije. Ovaj sustavni pregled obuhvaća njihovu analizu i odgovara na istraživačka pitanja o učestalosti primjene ovih metoda i svrsi njihove uporabe.

Ključne riječi: klasifikacijska stabla, osnovnoškolska matematika, regresijska stabla, rudarenje podataka, stabla odlučivanja

Hybrid Teaching and Outcome Indicators of the State Graduation Exam in Mathematics

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Abstract. Although the use of multimedia in terms of modernization in teaching process has been for some time a subject of study in the fields of pedagogy and didactics, the educational system in the Republic of Croatia had to engage in a specific way in the field of hybrid learning strategies in the past three years. Two devastating earthquakes and several waves of the advanced pandemic of the COVID-19 virus caused the need for alternative teaching models as opposed to classical and contact ones. In addition to the above, it required adaptation and realization of new competencies for both students and teachers, and even though it ensured uninterrupted education, it also pointed to certain problems and negative consequences. This paper provides an overview of the requirements, challenges, advantages and disadvantages of hybrid teaching, specifically showing the results of the state graduation exam in mathematics among students of Croatian schools, who took the exam after attending school in the mentioned conditions of the specific social situation. Recommendations for science and practice based on the indicated comparisons suggest the observation of the mentioned causes and related phenomena in the outcomes and success of teaching both in this and other school subjects and at all levels of upbringing and education.

Keywords: advantages and disadvantages of hybrid teaching, hybrid teaching in the Republic of Croatia, hybrid teaching of mathematics, results of the state graduation exam in the Republic of Croatia, teaching during the pandemic

1. Introduction

For the last two decades, there has been extensive research into the implementation of digital multimedia in the teaching process in the Republic of Croatia (Dumančić et al., 2013; Eret, 2015, 2016, 2018; Mačinko Kovač & Eret, 2012; Matasić et al., 2011; Matijević, 2015). Since this trend is in line with technological and cultural trends, it was expected that digital media might soon assume a specific role in the field of education and training. This has generated the interest of pedagogues and educators, who examined the correlation between digital media and student motivation and communication and socialization, and finally, learning and teaching outcomes. Some research indicated the benefits and drawbacks of multimedia-enriched teaching processes, while others examined the expectations of using multimedia in the coming years (Eret, 2016). However, what has not been anticipated is the element of the situation in which Croatia found itself, having undergone the trauma of two devastating earthquakes and the full-scale COVID-19 epidemic. The latter required citizens to isolate for a shorter or longer period and in the aftermath of said earthquakes, many residents were left without proper accommodation and working conditions. All this has challenged and tested the viability of education and training processes on all levels (Bokulić, 2021). Admittedly, the period of two to three weeks, which is how long it took for the country to devise teaching models and set up digital technologies to run them, was impressive. The entire teaching process became digital. This meant that teachers, as well as students, needed to adapt and grasp the concepts of digital technologies in a short amount of time. At first, success in that area was owed to every teaching professional and student, as well as to the entire schooling system, as it managed to endure and ensure the continuity of education and training (Bokulić, 2021; Golubić, 2021). However, now that the adverse times are behind us, and instruction is again in-person, we are starting to identify the shortcomings as well. The questions persist: Is the success of students receiving online instruction digitally different in terms of learning and teaching outcomes (Tonković et al., 2020)? Are state graduation exam results an indicator thereof (Golubić, 2021)? Given that in other fields problems brought on by hybrid teaching have arisen (connected to communication, socialization, and social networks), the issue deserves careful consideration (Tonković et al., 2020).

2. From traditional to digital and hybrid instruction

As digital media became a greater part of our everyday private and professional activities, scientists in the field of pedagogy, didactics, and methodology began to point out the ideas of implementing multimedia into the teaching process (Dumančić et al., 2013; Eret, 2015, 2016; Matijević, 2015). However, research in this area also examined whether this change facilitates a shift away from traditional instruction which was largely analog. Some scientists found that the initial research, which showed that digital media largely surpasses traditional teaching, is actually the consequence of students' great interest in the new digital "toys" in the classroom, rather than this being an indicator of higher teaching efficiency.

Later research indicated divisive results when comparing traditional and digital teaching, while some findings spoke in favor of the former (Clayton et al., 2010). Building upon scientific interests, the research looked into the indicators of the relationship between in-classroom multimedia and motivating students for a school subject (Clayton et al., 2010; Eret, 2016). Once again, apart from multimedia, some factors, such as the teacher, the quality of the teaching environment, and the teaching models, proved to be more relevant (Eret, 2015, 2016). In fact, one of the key hallmarks of the digital age, that is, hybrid teaching, is digital communication and social skills in general. In the beginning, the digital aspect presented a challenge for both teachers and students in terms of acquiring new competencies and mastering the use of different technologies (Eret, 2015, 2016). This is why smartphones, tablets, the Internet, email, chats and particularly social media became more present. In fact, social networks are a representation of how private means of communication became an alternative to teaching communication. In the past decade, video-conferencing platforms have become the omnipresent backbone of electronic meetings, as well as lessons being held in a hybrid manner (Skype, Zoom, MS Teams, etc.). Today, those tools are what helps us to carry out teaching and learning quickly, efficiently, and securely (Breslauer 2011; Eret, 2016; Matasić et al., 2011). Instead of asking if remote teaching has become better than in-person teaching, we should wonder which one is more needed. We can then distinguish between face-to-face teaching, out-of-class teaching, and digitally assisted teaching focused only on ensuring teaching materials digitally, without teaching. Research shows that hybrid teaching has the highest success rate because it encompasses all three said approaches. In fact, teaching process outcomes have been most met successfully when the hybrid teaching model was adopted according to Hutinski and Auer (2009). The hybrid model includes content in digital form, which is distributed online, teaching, and teacher-student communication, which is periodically carried out virtually. Nevertheless, education experts underline that teachers are irreplaceable, both in the classroom and beyond, as they create a teaching environment, connect with students, and facilitate connections among students themselves. This makes the traditional teaching method indispensable not just in the academic sense, but also as an element of socialization, which is why it represents the third pillar of the hybrid teaching model (Eret, 2015, 2016).

3. Indicators of shortcomings of some segments of hybrid teaching

In any case, there is an underlying assumption that in the hybrid model, it is not possible to ensure the same teaching environment as in the classroom (Eret, 2015, 2016). Teachers and students state that the key thing missing in remote teaching is the mutual interaction between them, and among students. They believe that interaction, which is the key to facilitating communication and socialization and ensuring and understanding the content and related tasks, is more difficult to achieve in an environment restricted to computer screens (Eret, 2015, 2016; Mačinko Kovač & Eret, 2012). Furthermore, previous research has shown that, especially when teaching was carried out exclusively remotely, digital media have brought about social and communicative alienation among students (Mouttapa et al., 2004). Some studies have focused on the link between digital media and students' motivation to acquire new information, finding that there were no statistically significant differences compared to traditional teaching (Clayton et al., 2010; Eret, 2016; Jukić, 2017). This paper examines the connection between digital media and students' success as seen in learning and teaching outcomes. Arguments put forward in this chapter are an incentive to examine whether a discrepancy exists in learning outcomes among senior students in the past three years compared to students graduating high school prior to that time. In any case, it is very unlikely that there is going to be a downward trend in the digitalization of teaching. It should therefore be examined if some elements of remote teaching are linked to poorer results in learning outcomes. Looking at those elements singled out, we gain a clearer perspective on how the hybrid model of teaching should be improved so as to ensure a more successful acquisition of learning outcomes in the coming generations of students.

4. Considerations of the role and importance of state graduation exams

The question that arises is how the "forced" remote learning brought on by the 2020 pandemic impacted students and their academic success. A lot could be studied about the social and communication skills, the psychological condition of all participants. Here, however, we shall focus on the success achieved on the state graduation exam and the link between that and, perhaps, the actual depiction of the situation in education at the said time.

The following tables comprise the results of the math exam at the basic and extended level of the state graduation exam from 2013-14 to 2020-21 school years (Bajrović, 2019; Nacionalni centar za vanjsko vrednovanje obrazovanja [NCVVO], 2019, 2020a, 2020b, 2021). As can be seen from the tables, at the extended level, the grade percentage ranges were lowered at one or several instances throughout the said periods. However, the ranges were increased in only one year at the extended level of the state graduation exams, and only for a single grade.

Table 1. Grade percentage ranges of the math exam, basic level of the state graduation exams, summer term (Bajrović, 2019, p. 6; NCVVO, 2019, 2020a, 2020b, 2021).

GRADE	2013./14.	2014./15.	2015./16.	2016./17.	2017./18.	2018./19.	2019./20.	2020./21.
1	0–24.99	0–24.99	0–24.99	0–23.99	0–24.99	0–24.99	0–24.99	0–24.99
2	25–44.99	25–49.99	25–49.99	24–46.99	25–46.99	25–46.99	25–44.99	25–44.99
3	45–64.99	50–69.99	50–69.99	47–66.99	47–66.99	47–66.99	45–66.99	45–66.99
4	65–82.49	70–86.99	70–86.99	67–84.99	67–84.99	67–84.99	67–84.99	67–84.99
5	82.5–100	87–100	87–100	85–100	85–100	85–100	85–100	85–100

Table 2. Table 2 Grade percentage ranges of the math exam, extended level of the state graduation exams, summer term (Bajrović, 2019, p. 7; NCVVO, 2019, 2020a, 2020b, 2021).

GRADE	2013./14.	2014./15.	2015./16.	2016./17.	2017./18.	2018./19.	2019./20.	2020./21.
1	0–28.32	0–27.99	0–27.99	0–24.99	0–24.99	0–24.99	0–24.99	0–24.99
2	28.33–49.99	28–49.99	28–49.99	25–45.99	25–45.99	25–44.99	25–44.99	25–44.99
3	50–71.66	50–69.99	50–69.99	46–65.99	46–67.99	45–67.99	45–67.99	45–67.99
4	71.67–86.66	70–82.99	70–82.99	66–80.99	68–84.99	68–84.99	68–84.99	68–84.99
5	86.7–100	83–100	83–100	81–100	85–100	85–100	85–100	85–100

Table 3. Distribution of grades of the math exam, basic level of the state graduation exams, summer term (NCVVO, 2019, 2020a, 2020b, 2021).

GRADE	2017./18.	2018./19.	2019./20.	2020./21.
1	15.90 %	20.70 %	15.80 %	8.00 %
2	35.40 %	44.30 %	33.14 %	31.60 %
3	27.20 %	26.30 %	36.11 %	36.40 %
4	15.10 %	8.00 %	12.90 %	18.00 %
5	6.40 %	0.70 %	2.05 %	5.90 %

Table 4. Distribution of grades of the math exam, extended level of the state graduation exams, summer term (NCVVO, 2019, 2020a, 2020b, 2021).

GRADE	2017./18.	2018./19.	2019./20.	2020./21.
1	11.90 %	20.60 %	13.34 %	11.00 %
2	37.90 %	37.40 %	34.38 %	26.70 %
3	30.30 %	28.30 %	32.53 %	37.00 %
4	13.10 %	9.60 %	13.64 %	16.40 %
5	8.80 %	4.10 %	6.11 %	8.90 %

The aggregate score of the basic and extended level math exam in 2017-18 (Bajrović, 2019) reveals that 51 % of the test-takers received a negative mark or a D, and only 21 % received a B or an A. Regardless of the fact that in the subsequent year, the grade percentage range was lowered by 1 %, the aggregate overall exam score was lower (cf. Table 4). If we disregard the fact that the grade percentage range was decreased by an additional 2 % at the basic level in 2019-20, we could deduce that in the year when the pandemic began, the students were somewhat more successful compared to the ones in the year before (cf. Table 3).

These statistical cases usually beg various questions. Why was the grade percentage range lowered and was this the reason behind seniors' higher success rate? How demanding was the test, on both levels, that year? Do the state graduate exam result and difficulty differ from year to year? How realistic is it in indicating student achievements as a statistic?

Such exam results are usually followed up in the media. For instance, in 2023, Dnevnik.hr (2023) wrote: “Seniors have produced great state graduation exam results – the overall average grade was a 4. More specifically, at the extended level, the seniors averaged a 3.5, and at the basic level a 2.6”. However, what they fail to mention is that the grade percentage ranges on the state graduation exam were in fact lower than in schools. Even if they did mention it, the media saw it as an improvement: “It is commendable that 19 students who completed the school year with an average grade of 2 achieved a grade 5 at the exam, and most of them in math” (Dnevnik.hr, 2023). It is never mentioned which school the seniors had graduated from so we are not aware if some went to grammar schools and ended up taking the basic level of the exam. This may turn out to be the answer to the question: How is it possible that a student with a D in math achieved an A on the state graduation exam, particularly in light of the fact that students scored the best results in math and English since the state graduation exam is being implemented (Glas Slavonije, 2023). This school year, state graduation exams can be taken by all high school students that wish to enroll in professional undergraduate studies, which was previously not the case. This is in accordance with Article 58 Paragraph 3 of the Act on Higher Education and Scientific Activity. It is not clear what results can be expected.

We are not wrong to question the objective criteria for awarding grades in schools. Does that mean that his final grade in mathematics was not a realistic representation of his ability? Does the comparison between school success and state graduation exam results unjustifiably tarnish the reputation of his elementary and high school teachers (Županović, 2018)? If there would be final exams for any school subject in elementary or secondary schools, can it be compared with and can it be projection of student’s effort and activity during entire school year (for that is what final grades stand for)? Accordingly, would these final exam scores relate to the ones from the initial exams of particular school subject at the beginning of a next school year? Possibly it may result in even greater discrepancies and disharmony between the final grade and the state graduation exam results.

Research done at the University of Rijeka found that since the state graduation exams are implemented, enrollment into the Faculty of Medicine became easier for grammar school applicants. This is because the points assigned for their high school grades are taken into greater consideration compared to the applicants coming from a medical high school. The latter then have a smaller chance of enrolling in medical studies, since the physics, chemistry and biology curricula of medical high schools cannot compete with the ones of grammar schools (Žauhar et al., 2016). This begs the following questions: Is the state graduation exam, sometimes, an end in itself? Is it fair to assess teachers based on the results their students achieve on those exams? Can conclusions be drawn similarly for elective and mandatory subjects? In elementary schools, national exams were introduced for students completing the 8^{th} grade so as to test the skills and competencies they have acquired throughout primary education. This has been done to get a realistic view of elementary school students' achievements individually, at the level of the school, and on the national level (Filipović, 2022). While this remains vital, it is still unclear whether students can achieve the desired goal, knowing in advance that their national exam result will ultimately not lead to punishment or a reward. This brings into question the objectivity of conclusions made based on the results of the national exams. Therefore, it is questionable whether national exams may become an additional indicator of the success of students and the education system itself.

5. Conclusion

Several dilemmas that have arisen from this paper and concern science and practice should be addressed in the near future. More specifically, the uncertainties are directly related to the improvement of Croatian education and the expectations for future generations. They also concern the role, significance, and purposefulness of state graduation exams (Bajrović, 2019; Žauhar et al., 2016; Županović, 2018). What also must be reexamined is the link between the Croatian schooling system and the success of the state *matura* (Jukić, 2017; Tonković et al., 2020). The backbone of this paper is to reconsider whether indicators of the success of state graduation exams in Croatia are correlated with the situational elements in education in general. If so, these have to be detected and action must be taken to address the negative indicators. Possibly it refers to the hybrid teaching which has, for the most part, preceded the state exams in the past two years (Bokulić, 2021; Golubić, 2021). The hybrid model has its advantages and disadvantages (Clayton et al., 2010; Eret, 2015, 2016; Hutinski & Aurer, 2009). Nevertheless, we cannot claim that the situation in which Croatian education finds itself is merely a reflection of the teaching model which largely relies on digital media due to the fact that they have been useful in overcoming sudden and drastic societal (and schooling) changes. However, what remains to be considered is how to purposefully approach the digitalization of teaching and assessing student achievements upon completion of education. It is also worth evaluating how much that assessment is harmonized with the overall education system of the country, particularly with regard to assessing learning outcomes and teaching throughout the teaching process.

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Hibridno izvođenje nastave i pokazatelji ishoda državne mature iz matematike

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Sažetak. Iako je uporaba multimedija u svrhu osuvremenjivanja nastave već određeno vrijeme predmetom proučavanja na poljima pedagogije i didaktike, odgojno-obrazovni sustav u Republici Hrvatskoj se u protekle tri godine na specifičan način morao angažirati na polju strategija hibridnog učenja. Dva razorna potresa i nekoliko valova uznapredovale pandemije virusa COVID-19 uzrokovali su potrebu alternativnih modela nastave nasuprot klasičnim i kontaktnim. Osim što je navedeno zahtijevalo potrebe prilagodbe i ostvarivanja novih kompetencija kako učenika tako i nastavnika i iako je osiguralo neprekidanje tijeka školovanja, ukazalo je i na određene probleme i negativne posljedice. Ovaj rad donosi prikaz zahtjeva, izazova, prednosti i nedostataka hibridne nastave, konkretno prikazujući rezultate ishoda državne mature iz matematike među učenicima hrvatskih škola, a kojima je neposredno pisanju ispita prethodila nastava u navedenim uvjetima specifične društveno-socijalne situacije. Preporuke za znanost i praksu na temelju ukazanih usporedbi sugeriraju i promatranje navedenih uzroka i povezanih pojava u ishodima i uspješnosti nastave kako u ovom, tako i drugim nastavnim predmetima na svim razinama odgoja i obrazovanja.

Ključne riječi: hibridna nastava matematike, hibridna nastava u Republici Hrvatskoj, ishodi državne mature u Republici Hrvatskoj, nastava u vrijeme pandemije, prednosti i nedostaci hibridne nastavea

Analysing Students' Competence in Basic Logical Operations and Logical Reasoning

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Abstract. Empirical research results show that the system of logical operations and reasoning is not complete by adulthood. School or university curriculum content cannot be successfully mastered without the use of key logical chains of reasoning. Many subjects require proofs, reasoning and refutation, and scientific terminology uses a large number of logical operations. Our 2022 end-of-year research, carried out on 246 students, tested basic general logical reasoning skills and operations in the context of mathematics, physics, chemistry and biology. Calculating reliability using the Kuder-Richardson formula 21, we found that the test is reliable and its results are consistent. Students achieved the best results in the tasks that required logical reasoning (69.02 %); understanding the concepts 'at most/at least' (63.41 %) came second, and they were least successful with tasks requiring negation (the negation of 'at least', 'there is' and 'all') (29.91 %). The correct reasoning was not dominant in the case of all logical operations. The highest success rate of logical operations was achieved in mathematics (60.27 %), followed by biology and physics (58.73 %), while chemistry was lagging behind (38.71 %). Correlation analysis shows that there is no significant difference between the results of different majors, between students in the bachelor or master's programmes or between genders.

Keywords: basic knowledge of mathematics, logical operations, transfer of knowledge, mathematical abilities, problem solving abilities

1. Introduction

Higher education curriculum builds on knowledge acquired in high school. Studies show that students pursuing higher education are at different levels of preparation. The ability to apply mathematical knowledge is essential. Practical knowledge is often separated from theoretical knowledge. Students find it difficult to bind these two or to see the relationship between them.

One of the main features of the Romanian educational system is that students can study Mathematics at different levels. A graduate of a Mathematics-Computer Programming specialization at a public school may even take a Baccalaureate examination of high mathematical knowledge as the programme includes: derivatives, probability theory, the basics of linear algebra, complex numbers, etc.

Studies have shown that knowledge transfer is not automatic; the acquired knowledge is not immediately carried over to novel situations (Csapó, 1998).

Applying the acquired knowledge in novel situations indicates a different quality of knowledge, a deeper understanding. The aim of this research was to test students' independent thinking and problem-solving skills, as well as to investigate the way they can actively apply their knowledge when solving problems not directly related to the curriculum. Meaningful learning, acquisition, and understanding are basic aspects of all types of learning. Examining the different layers of understanding and knowledge is not an easy endeavour. We asked 246 students to take a test that contained problems involving logical operations, which required deeper understanding. This makes the test suitable for analysing useable knowledge.

International assessments in science and mathematics carried out during the 1970s and 1980s were curriculum based and investigated how students acquire disciplinary knowledge and how they apply it in a context similar to that encountered in the classroom.

The OECD PISA (Programme for International Student Assessment) focuses on three knowledge areas (reading, mathematics and science) assessing students' applicable knowledge, which is essential for a full participation in a modern society (assessing the application of knowledge in novel situations).

Complex problem solving, as the fourth domain covered in PISA 2003, opened up a new dimension in international assessment: assessing general reasoning skills not directly related to school subjects (Tóth et al., 2010).

There is a large number of studies on the question of whether and to what extent the mathematical knowledge acquired at school is incorporated into everyday knowledge and contributes to the development of reasoning. The literature points out that transfer is not automatic, the acquired knowledge cannot be automatically transferred to novel situations (Csapó, 1998).

2. Theoretical background

Logical operations are already taught in elementary school. At school we are mainly taught how to reason, which is why it is important to be familiar with logic. If we want children to be able to make correct and true statements in all subjects, to use concepts correctly and make correct deductions, we must make sure to use concrete examples that are consistent with the principles of bivalent logic (Olosz & Olosz, 1990).

In a study involving 7th and 11th grade children, Vidákovich (1998) investigated the development of deductive reasoning. He pointed out that the development of logical operations, alongside a relatively modest increase in overall performance, is rather a kind of structured rearrangement. Reasoning, equivalence and implication were among the least developed operations. The researcher also points out that the system of logical operations and reasoning is characterized by a slow development which does not reach completion by adulthood.

There is a large number of studies on the question of whether and to what extent the mathematical knowledge acquired at school is incorporated into everyday knowledge and contributes to the development of reasoning. The literature points out that transfer is not automatic, the acquired knowledge cannot be automatically transferred to novel situations (Csapó, 1998).

Kézi (2020) conducted an experiment with two groups consisting of 12th grade students. For one of the groups, the tasks were formulated in the "classical mathematics" style, while the same tasks were formulated in an application-oriented style for the other group. Results showed that the "purely mathematical" tasks were solved correctly by a larger percentage than the application-oriented tasks.

A Nigerian study focuses on the use of logical operations, on the if-then statement in particular. According to the author, challenging mathematical problems are not only conductive to a more accurate mathematical knowledge and better mathematical skills, but they also develop logical thinking that is necessary not only to people in the field of mathematics, but to everyone who wants to become a useful member of society (Salami, 2021).

We chose university students as our research sample based on the studies presented above and in addition to purely mathematical statements, we also formulated tasks in the context of chemistry, biology and physics.

3. The research

The research methodology involved the completion of a test by 246 students majoring in economics, teacher training, engineering and various other subjects (Table 1): 179 undergraduates (72.76 %) and 67 master's students (27.23 %). 196 (79.7 %) participants were from Partium Christian University, while 50 (20.30 %) participants were from the Satu Mare extension of BBU. Distribution of students according to gender: 187 female (76.01 %) and 59 male (23.98 %).

Distribution according to age: 217 participants aged 18–25 (88.21 %), 14 participants aged 26–35 (5.69 %), 14 participants aged 36–45 (5.69 %) and one participant aged 45+ (0.40 %).

The data were collected in December 2022.

Table 1. Distribution of Sample According to Major.

Major	Individuals
Economics (Bachelor's degree)	77 (31.30 %)
Mechanical Engineer (Bachelor's degree)	9 (3.65 %)
Pedagogy of primary and preschool education (Bachelor's degree)	93 (37.80 %)
Economics (Master's degree)	27 (10.97 %)
Teacher training (mixed: music, languages, social studies, art) (Master's degree)	40 (16.26 %)
Total	246

As a research tool, we used an online questionnaire developed by a research group from Slovakia

(Szarka et al., 2021, 2022). The test contained a total of 15 items divided in three problem sets (A, B and C). The sets contained the following type of problems:

Set A: understanding the concepts 'at most/at least'.

Set B: negation: at least, there is, all.

Set C: reasoning: if...then, therefore.

By means of these problems we investigated the background of logical operations, students understanding of basic concepts and the use of these in everyday life. The tasks tested the skills components of knowledge, the understanding of the text, deeper knowledge and the secure integration of this skills into the students' overall knowledge system. A significant aspect when selecting the problems was that they should not be rich in mathematical content. We chose problems which we deemed essential given their applicability to other subjects or fields. All three problem sets contained **problems of the same type** (understanding the concepts 'at most/at least', negation: at least, there is, all, and logical reasoning: if... then, therefore) but presented in **different contexts**: we tested basic general logical reasoning skills and operations in the context of mathematics, physics, chemistry and biology.

Students were given thirty minutes to complete the test. Answers, i.e. each item, were rated on a dichotomous scale (right/wrong). Students scored 1 point for correct answers and 0 points for wrong answers.

The tasks were of nearly the same difficulty. The KR21 is a special case of Cronbach's Alpha in which the items are binary variables, usually scored as 0 or 1, and the tasks are of nearly the same difficulty. The Kuder–Richardson formulas, first published in 1937, are a measure of internal consistency reliability for measures with dichotomous choices (Kuder & Richardson, 1937).

The KR-21 formula is: $n/(n-1) \cdot [1-(M \cdot (n-M)/(n \cdot Var))]$, where n is the sample size, Var is the variance, M is the mean score for the test. The value for KR-21 ranges from 0 to 1, with higher values indicating higher reliability. Based on the Kuder-Richardson formula 21 our test is reliable, our value is near 0,7, the results are consistent.

3.1. The test used for the research

Set A

A1. To pass the test one must score at least 95 points out of a total 120 points. For which of the following scores was the candidate unsuccessful?
a) 94 b) 95 c) 96 d) 120

A2. If the lift has a load of more than 240 kg, the "overloaded" indicator lights up. Four people got into the lift. How much can they weigh at most?
a) 238 kg b) 240 kg c) 239 kg d) 241 kg

A3. The solution to an inequality is a set of integers at least -2 and at most 2. How many solutions are there to the inequality?
a) 5 b) 4 c) 3 d) 2

A4. Chlorine compounds most often have an oxidation state of an odd number between -I and VII. Which of the following statements is true?
a) Chlorine compounds can occur in 5 oxidation states at most.
b) Chlorine compounds can occur in 7 oxidation states at most.
c) Chlorine compounds can occur in at least 5 oxidation states.
d) Chlorine compounds can occur in at least 7 oxidation states.

A5. The four letters of the RNA chain are coded in triplets. One triplet encodes one amino acid. At least how many amino acids can be found in the following RNA chain?
AAU GCA CCA AGA CGG GAA CUU GAU GAA C
a) 9 amino acids b) l0 amino acids c) 8 amino acids d) 7 amino acids

Set B

B1. "*There are heat receptors that are found not only in the skin.*" The negation of this statement is:
a) There are no heat receptors in the skin.
b) All heat receptors are in the skin.
c) Some heat receptors are not in the skin.
d) Every receptor in the skin is a heat receptor.

B2. Negate the following statement. "*I have been to Prague at least 6 times so far.*"
a) I have been to Prague 6 times so far.
b) I have been to Prague 6 times at most.
c) I have been to Prague at least 5 times so far.
d) I have been to Prague 5 times at most.

B3. "*Liquids (e.g. water) evaporate at all temperatures.*" The negation of this statement is:
a) There is a liquid that does not evaporate at all temperatures.
b) No liquid evaporates at all temperatures.
c) There is a temperature at which liquids do not evaporate.
d) There is no such temperature at which liquids evaporate.

B4. According to the current state of sciences, under standard conditions: "*2 of the elements in the periodic table are in liquid state*". Choose the correct negation of the statement!
a) at least 3 elements of the periodic table are in liquid state
b) 2 or more elements of the periodic table are in liquid state
c) at most 1 element of the periodic table is in liquid state
d) at most 1 or at least 3 of the elements of the periodic table are in liquid state
e) at most 1 and at least 3 of the elements of the periodic table are in liquid state

B5. Negate the following statement: "*The sum of the interior angles of all rhombuses is 360 degrees.*"
a) The sum of the interior angles of all rhombuses is 359 degrees.
b) There is a rhombus, in the case of which the sum of the interior angles is not 360 degrees.
c) The sum of the interior angles of all rhombuses is 359 degrees at most.
d) The sum of the interior angles of a rhombus is not 360 degrees.

Set C

C1. If the pH is greater than 7, the solution is basic. When dissolved in water, the pH of shower gel is 5, the pH of baking soda is 9 and the pH of washing powder is 12. Based on the information above, which statement is FALSE?
a) Shower gel and washing powder are basic.
b) Shower gel or washing powder are basic.
c) Baking soda and washing powder are basic.
d) Baking soda or washing powder are basic.
e) At least one of the three solutions is basic.

C2. A natural number is divisible by 6 only if it is divisible by 2 and 3. I thought of a number that is divisible by 2. Then...
a) ... this number is divisible by 6.
b) ... this number is divisible by 3, but it is not divisible by 6.
c) ... this number is not divisible by 6.
d) ... it is not possible to decide whether this number is divisible by 6.

C3. The speed of sound in a medium is determined by the density of the medium. A sound wave will travel faster in a more dense material. The density of iron is greater than the density of air, therefore...
a) sound travels slower in iron
b) sound travels faster in air
c) sound travels faster in iron
d) sound travels at the same speed in iron and air

C4. All mathematicians like cats and dogs. Tomi is a mathematician. Based on the information above, which statement is correct?
a) Tomi likes cats but he doesn't like dogs.
b) Tomi likes only dogs.
c) Tomi likes both cats and dogs.
d) Tomi likes neither cats nor dogs.

C5. The smaller a bird, the more eggs it lays. The eagle is a large bird, therefore.... (continue the reasoning)

The solutions to the test are given below (see Table 2).

Table 2. Key for the Test.

Problem set A	correct	Problem set B	correct	Problem set C	correct
A1 – test	a	B1 – heat receptor	b	C1 – basic	a
A2 – lift	b	B2 – Prague	d	C2 – divisibility	d
A3 – inequality	a	B3 – evaporation	c	C3 – speed of sound	c
A4 – chlorine	c	B4 – periodic t.	d	C4 – Tomi	c
A5 – RNS chain	a	B5 – rhombus	b	C5 – eagle	word problem

For C5: acceptable answers: it lays fewer eggs, it lays (relatively) few eggs not acceptable: it lays several/many eggs, does not lay eggs, it lays 1 egg, it lays one or two eggs – and similar answers

3.2. Research questions

We posed the following research questions:

1. For each problem set, what percentage of students complete the tasks?
2. In which context do students perform better in logical operations: mathematics, biology, physics or chemistry? Are there any significant differences?
3. Is there a significant difference between the answers of different majors?
4. Is there a significant difference between the number of good answers achieved by undergraduate and master's students?
5. Is there a significant difference between the answers of students of different gender?
6. Is there a significant difference between the answers of students of different age?

3.3. Hypotheses

The following hypotheses were formulated:

1. Each problem set will yield different results.
2. The proportion of correct answers will depend on the context in which the task is formulated.
3. Economics and engineering students will perform better than students in teacher training programs.
4. Master's students will perform better than undergraduates.
5. Male respondents will perform better than female respondents.
6. Older students will perform better than younger students.

4. The results of the survey

For the first research question, students completed the tasks in the following percentage for the different problem sets (Table 3):

Table 3. Percentage of Correct Answers and Average Performance per Problem Set, Adjusted Standard Deviations.

Problem set	**A (A1, A2, A3, A4, A5)**	**B (B1, B2, B3, B4, B5)**	**C (C1, C2, C3, C4, C5)**	**Overall results**
Total ($N = 246$)	63.41 % (780 correct answers out of 1230)	29.91 % (368 correct answers out of 1230)	69.02 % (849 correct answers out of 1230)	54.11 % (1997 correct answers out of 3690)
Average (from a maximum of 5 points)	3.17	1.49	3.45	2.70
Adjusted standard deviation	1.00	1.12	1.15	1.096

Students achieved the best results in problem set C (reasoning: if... then, therefore) (69.02 %), followed by set A (understanding the concepts 'at most/at least'), with a percentage of 63.41 % correct answers. They were least successful in problem set B (negation of 'at least', 'there is' and 'all'), with a percentage of 29.91 % correctly solved problems.

The Single-factor ANOVA shows that there is a significant difference in the success rate of solving different tasks ($F = 228, 83$, $p \leq 0, 001$), and negation of different statements including quantifiers as 'every/everyone', 'at least', 'there are' is the most difficult logical operation for students. The most difficult task was to negate the sentence with 'at least'. Although, according to the rules of formal logic, a relatively simple situation from everyday life had to be negated ('I have been to Prague at least 6 times'), from 246 participants a total of 36 correct answers were received, giving a 14.63 % success rate. The same low percentage of students (19.92 %) negated correctly the sentence '2 of the elements in the periodic table are in the liquid state' in a chemistry context.

The study also aimed to identify the context in which students achieved the highest results with logical operations: mathematics, biology, physics or chemistry.

Results are shown below (see Table 4).

Students achieved the best results in the context of mathematics (60.27 %), followed by biology and physics (58.73 %) (see Table 4). They were least successful with the chemistry set, achieving a percentage of 38.71 % correctly solved problems. All three sets contained problems of the same type (understanding the concepts 'at most/at least', negation: at least, there is, all, and logical reasoning: if... then, therefore) but in different contexts.

Table 4. Percentage of Correct Answers and Average Performance per Subjects, Adjusted Standard Deviations.

Problem set	**Mathematics (logics) (A1, A2, A3, B2, B5, C2, C4)**	**Chemistry (A4, A5, B4, C1)**	**Biology, physics (B1, B3, C3, C5)**
Total ($N = 246$)	60.27 % (1038 correct answers out of 1722)	38.71 % (381 correct answers out of 984)	58.73 % (578 correct answers out of 984)
Average	4.21 (from a maximum of 7 points)	1.54 (from a maximum of 4 points)	2.34 (from a maximum of 4 points)
Adjusted standard deviation	1.2	0.96	1.00

The third question posed was whether there is a correlation between the percentage of correct answers and the majors.

Table 5 shows the distribution of the observed values (number of correct answers) by major and by problem set.

Table 5. Percentage of Correct Answers by Major per Problem Set.

Major	**A**	**B**	**C**	**Total**
Economics (BSc)	250 (64.93 %)	114 (29.61 %)	262 (68.05 %)	626 (54.19 %)
Mechanical Engineer (BSc)	32 (71.11 %)	11 (24.44 %)	27 (60 %)	70 (51.85 %)
Pedagogy of primary and preschool education (BA)	302 (64.94 %)	127 (27.31 %)	321 (69.03 %)	750 (53.76 %)
Economics (MSc)	73 (54.07 %)	48 (35.55 %)	94 (69.62 %)	215 (53.08 %)
Teacher training (mixed: music, languages, social studies, art) (MA/MSc)	123 (61.5 %)	68 (34 %)	145 (72.5 %)	336 (56 %)
Total	780 (63.41 %)	368 (29.91 %)	849 (69.02 %)	1997 (54.11 %)

In set A, mechanical engineering majors provided the most correct answers (71.11 %), while the smallest number of correct answers were given by economics majors (54.07 %). In set B, economics master's students achieved the highest percentage of correct answers (34 %), while mechanical engineering undergraduates were the least successful (24.44 %). In set C, teacher training students achieved the best results (72.5 %) while mechanical engineering majors performed the least well (60 %) (see Table 5).

Correlation analysis between the majors on different fields and the number of correct answers provided in different logical operations type A, B, C: the chi-square test yielded no significant difference. ($P = 0.52$ there is no significant correlation.)

Considering the results by majors, it can be concluded that BSc economics students performed best in solving the tasks in set C; mechanical engineering undergraduates performed best in set A, students enrolled in the primary and preschool teacher training program were most successful in set C, economics master's students performed best in set C, and so did teacher training students. Set B, which tested negation, seemed to be more difficult for all groups of students and few good answers were given.

Table 6 presents the results (the number of correct answers) of undergraduates and master's students achieved in the test.

Table 6. Number of Correct Answers per Problem Set for Undergraduates and Master's Students.

Students	**A**	**B**	**C**	**Total**
BSc/BA	584 65.25 %	252 28.15 %	610 68.15 %	1446 53.85 %
MSc/MA	196 58.50 %	116 34.62 %	239 71.34 %	551 54.82 %
Total	780 63.41 %	368 29.91 %	849 69.02 %	1997 54.11 %

Table 6 above shows that undergraduates performed better in set A, while master's students performed better in set B and C, however, no significant difference was found between the number of correct answers achieved on the test.

Another question we aimed to answer is whether there is a significant difference between the answers of students based on gender.

Table 7 below presents the results (the number of correct answers) by gender for the three problem sets.

Table 7. Distribution of the Number of Correct Answers by Gender and per Problem Set.

Students	**A**	**B**	**C**	**Total**
Male	180 61.01 %	94 31.86 %	171 57.96 %	445 50.28 %
Female	600 64.17 %	274 29.30 %	678 72.51 %	1552 55.2 %
Total	780 63.41 %	368 29.91 %	849 69.02 %	1997 54.11 %

Table 7 above shows that women performed better in set A and C, while men were more successful in providing the correct answers in set B.

As for the last research question, whether there is a significant difference between the answers of students of different ages, no significant difference was found. However, students performed better in set C, except for the one student who is over 45. Looking at each set, we found that the youngest age group had the best results in set A, with 64.7 % correct answers, the 26–35 year olds achieved best in set B, with 42.85 % correct answers, and the 26–35 year olds were most successful in set

C, with 75.71 % correct answers. Overall, we found that students aged between 26 and 35 achieved the highest number of correct answers, as shown in Table 8 below.

Table 8. Distribution of the Number of Correct Answers by Age per Problem Set.

Age	**A**	**B**	**C**	**Total**
between 18–25	702 64.7 %	319 29.40 %	744 68.57 %	1765 54.22 %
between 26–35	40 57.14 %	30 42.85 %	53 75.71 %	123 58.57 %
between 36–45	37 52.85 %	17 24.28 %	51 72.85 %	105 50 %
over 45	1 20 %	2 40 %	1 20 %	4 26.66 %
Total	780 59.09 %	368 27.87 %	849 64.31 %	1997 50.42 %

The percentage of correct answers per task is shown in Table 9 below. Students found questions A1 and C4 the easiest, and questions B2 and B4 the most difficult.

Table 9. The Number and Proportion of Correct Answers per Tasks and Problem Sets.

Task	**Number of correct answers**	**Percentage of correct answers (%)**
A1	242	98.37
A2	153	62.19
A3	139	56.50
A4	72	29.26
A5	174	70.73
A Total	**780**	**63.41**
B1	100	40.65
B2	36	14.63
B3	87	35.36
B4	49	19.91
B5	96	39.02
B total	**368**	**29.91**
C1	86	34.95
C2	133	54.06
C3	179	72.76
C4	239	97.15
C5	212	86.17
C total	**849**	**69.02**
Total	***1997***	***54.11***

5. Discussion

Discussing the findings in the light of the hypotheses:

1. Each problem set will yield different results.

Our first hypothesis was confirmed. Students performed best in set C, with a percentage of 69.02 % correct answers. Set A followed slightly behind with a percentage of 63.41 %, while the lowest percentage was registered for set B (29.91 %). The Single-factor ANOVA shows that there is a significant difference in the success rate of solving different tasks ($F = 228,83$, $p \leq 0,001$), and negation of different statements including quantifiers as 'every/everyone', 'at least', 'there are' is the most difficult logical operation for students.

2. The proportion of correct answers will depend on the context in which the task is formulated.

The hypothesis was confirmed as the highest percentage of correct answers was obtained for the mathematical (logic) problems (60.27 %), followed by slightly less correct answers for the biology and physics problems (58.73 %), while chemistry problems obtained the lowest number of correct answers (38.71 %).

3. Economics and engineering students will perform better than students in teacher training programs.

The hypothesis was not confirmed. There was no significant difference between the results of different majors. Teacher training students achieved the best results, with 56 % of the questions answered correctly, while mechanical engineering (BSc) students did the least well, with 51.85 % of the questions answered correctly.

4. Master's students will perform better than undergraduates.

Undergraduates achieved 53.85 % on the test, while master's students achieved 54.82 %. Nonetheless, no significant difference was found.

5. Male respondents will perform better than female respondents.

Male students achieved 50.28 % on the test, while female students achieved 55.2 %. The difference is not significant. The only significant difference between male and female respondents was in task 4 in set C, with 94.9 % of males and 97.9 % of females giving correct answers.

6. Older students will perform better than younger students.

Our hypothesis was confirmed as students aged 26–35 provided the highest number of correct answers. However, there is no significant difference between the results of the four age groups.

The significance of our study lies in the skill assessment methodology. Despite the critical role of basic logical operations in higher education for fostering critical and scientific thinking, there is a notable gap in literature concerning university students' proficiency in these skills. Through our conceptual replication study, we focused on the comprehension of the relationship between logical operations and context and the factors influencing the outcomes we observed.

Our results, largely in alignment with previous Slovak studies by Szarka et al. (2021, 2022), highlight that both the types of logical operations and the task contexts significantly impact students' performance. We found similar patterns across countries, indicating that performance is influenced more by the nature of the logical operations and task context than by differences in educational systems.

Specifically, our findings echo prior research suggesting that understanding quantifiers and particularly negation of quantifiers pose significant challenges for students (Dubinsky & Yiparaki, 2000; Ferrari, 2004; Hazem, 2017). This difficulty arises because some quantifiers require formal language comprehension, which students typically encounter only in higher education, leading to unfamiliarity with formalizations and visualization techniques crucial for logical tasks.

The struggles students face with these logical operations and conceptual comprehension may stem from the complexity of epistemology or inadequate emphasis on mathematical language in classrooms. Studies by Mesnil (2017), Ferrari (2004), and Bardelle (2013) emphasize the importance of distinguishing between everyday language and mathematical terminology to avoid confusion and errors in logical reasoning tasks.

In essence, our study sheds light on the nuanced challenges students encounter in mastering logical operations, emphasizing the need for greater awareness and support in integrating formal language and mathematical concepts into educational curricula.

One limitation of our study is the relatively small sample of students, drawn from universities in the peripheral border region, which may not fully represent the entire country. Comparing our students' performance to those in Slovakia (Szarka et al., 2021, 2022), our findings align with the consistently lower scores of Romanian students in PISA surveys, highlighting a disparity with the OECD average across all areas.

The mathematics curriculum, though aiming to foster logical thinking, differs in its approach between countries. While Slovakia explicitly incorporates logic, reasoning, and proof as separate areas, Romania embeds elements of logic within other topics without explicit inclusion. Additionally, pedagogical practices and teacher training systems vary, with Romania placing more emphasis on scientific disciplinary training over pedagogical-psychological subjects. These factors collectively influence students' performance on logical tests.

Our results align with prior research indicating no significant difference in performance between boys and girls in mathematical and logical tasks (Ramírez-Uclés & Ramírez-Uclés, 2020). Notably, refutation or negation of quantitative determinants emerges as one of the most challenging logical operations across contexts. The varying performance across different contexts, such as difficulty in solving chemistry problems, suggests that students' logical reasoning is context-dependent, underscoring the need for interventions to enhance their abstract logical inference abilities.

Given that a significant portion of our sample comprises students from preschool and primary education programs, nurturing their elementary reasoning skills is crucial not only for academic development but also for their future endeavors. Our findings advocate for the introduction of a dedicated subject aimed at practicing and honing basic logical operation skills and logical reasoning abilities.

Future research could be carried out to investigate whether there is a correlation between correct answers given and the evaluation of the difficulty of the task. If someone considers the task difficult, they cannot provide a correct answer, or what percentage of students who consider the task easy provides a wrong answer.

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Alapvető logikai műveleti készségek és a logikai érvelés vizsgálata egyetemi hallgatók körében

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Absztrakt. Az empirikus kutatási eredmények azt mutatják, hogy a logikai műveleti és következtetési sémák, ennek rendszere a felnőttkorra sem válik teljessé. Az iskolai, illetve egyetemi tananyag megértése a legfontosabb következtetési láncok használata nélkül nem lehet sikeres, a bizonyítások, az érvelés, a cáfolás gondolatmenetei szükségesek több tantárgy esetén, a tudományos szaknyelv nagy számban alkalmaz logikai műveleteket. A 2022-es év végi kutatásunk egy 246 fős hallgatói mintán általános logikai alapismereteket és műveleteket mért matematikai, fizikai, kémiai, illetve biológiai szövegkörnyezetben. A Kuder-Richardson 21-es formulával mérve a tesztünk megbízható, az eredményei konzisztensek. A hallgatók a legnagyobb arányban a helyes következtetést igénylő feladatokat teljesítették (69,02 %-ban), ezt követi a legfeljebb/legalább értelmezése (63,41 %), illetve legkevésbé a tagadást (a legalább, a létezik, a minden tagadását) (29,91 %). Elmondható, hogy nem minden logikai művelet esetében vált a helyes értelmezés dominánssá ezeknél a hallgatóknál. A hallgatók a legnagyobb arányban a logikai műveleteket matematika környezetben teljesítették (60,27 %), ezt követi a biológia, fizika (58,73 %), legkevésbé a kémiában tudják alkalmazni (38,71 %). Összefüggésvizsgálatokat végezve azt kaptuk, hogy nincs szignifikáns különbség a különböző szakon tanuló hallgatók helyes válaszai között, az alap-, illetve a mesterképzésben lévő hallgatók teszten elért jó válaszainak száma között, illetve a nemek között.

Kulcsszavak: matematikai alapismeretek, logikai műveletek, logikai következtetés, tudástranszfer

Mathematics Teachers' Continuous Professional Development – Reflections on Lifelong Learning Program "Enactive Learning in Mathematics"

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Abstract. The Lifelong Learning Program "Enactive Learning in Mathematics" was implemented for the first time at the Faculty of Mathematics, University of Rijeka, in the winter semester 2022/2023. The program enables participants to develop knowledge and skills to implement an enactive teaching and learning methodology adapted to the context of digital literacy that helps make mathematics more attractive to students, and guide students in the creation of enactive materials that support their learning with household objects. In addition, participants in this program experienced both roles, student and teacher, in an enactive mathematical learning. After the completion of the program, interviews were conducted with mathematics teachers who had participated in the program. They emphasized the benefits that the experience of program participation brought to their daily work, as well as the fact that they would implement some of the examples presented in the program by instructors or other program participants. In addition to the high applicability of the content in the mathematics classroom, the detailed instructions on the tasks, the clear structure of the program, the opportunity to exchange ideas, and the collegial advice from other mathematics teachers in a friendly and collegial environment are recognized as the main strengths of the program, an impact that is difficult to achieve in short professional development trainings. We also interviewed teachers about their motivation to participate in lifelong learning programs and the support they receive. This paper provides reflections on the interviews conducted and the program implemented. Based on the reflections of the Lifelong Learning Program "Enactive learning in Mathematics", we argue for a stronger presence of continuous professional development for mathematics teachers in Croatia.

Keywords: professional development, mathematics teacher, lifelong learning program, enactive learning in mathematics, education policy

1. Introduction

Enactive learning refers to the implementation of activities, experiments, and the concrete use of materials when introducing new mathematical content, mentally presenting mathematical content, and discovering of mathematical relationships.

Bruner assumed that learning a topic or an object of knowledge involves three nearly simultaneous processes: the acquisition of new knowledge, the transformation, and the evaluation of knowledge (Wen, 2018). He developed a learning theory based on a model that includes three modes of representation: enactive, iconic, and symbolic. These "*enactive-iconic-symbolic*" modes are not structures but different forms of cognitive processing (Schunk, 2021). Students understand their environment best when they interact with the objects around them, so a variety of materials should be used to represent mathematical concepts whenever possible. An enactive approach to learning mathematical content helps students build a mental network to understand and recognize mathematical concepts and relationships. Learning mathematics based on enactive (physically performed) metaphorization can significantly help reduce the cognitive abuse that millions of children worldwide face when exposed to mathematics (Soto-Andrade, 2018). Among other things, using the *Concrete-Pictorial-Abstract* (CPA) approach, which is a frequently used adaptation of Bruner's concept of "*enactive-iconic-symbolic*" modes of representation (see, e.g., Leong et al., 2015), can improve and develop elementary students' spatial skills (Putri et al., 2020), as well as students' performance in college algebra (Isip, 2018). Overall, enactive methods help to increase the understanding and appeal of mathematics and reduce learner underachievement based on lack of understanding. Therefore, the enactive approach is recognized and implemented by many mathematics teachers.

The implementation of the enactive approach in mathematics classes requires the use of different didactic materials and often the presence of teachers as facilitators of the research process carried out by the students. The didactic materials used are mostly prepared by the teachers and sometimes produced or obtained by the students according to the instructions they have previously received. The occurrence of the coronavirus pandemic had a significant impact on the teaching process in general and was particularly reflected in the lessons that involved the use of didactic materials available to students in their schools. Not only could students not come to school to have direct contact with their teachers and use the didactic materials they had prepared, but the ability to acquire and thus independently produce appropriate didactic materials was significantly reduced. In response to the coronavirus pandemic, two extraordinary Erasmus+ calls were issued by European Commission in August 2020, including the call on digital education readiness, which aimed to enhance online, distance and blended learning, including supporting teachers and trainers. (EC, 2020). The "Enactive Learning in Mathematics at Home (EnLeMaH)" project was one of the accepted projects under this call. The EnLeMaH consortium, coordinated by Bielefeld University (Germany), brings together four institutions from Germany, Croatia (University of Rijeka), Lithuania (St. Ignatius of Loyola University of Applied Sciences) and Spain (INCOMA).

One of the consequences of the EnLeMaH project is the development of the Lifelong Learning Program (LLP) “Enactive Learning in Mathematics” for mathematics teachers, which was approved by the University of Rijeka, and implemented for the first time in the winter semester of the academic year 2022/2023. During the implementation of the program, we noticed great enthusiasm and extremely positive reactions from the participants, which led us to explore what is the main reason for such a great success of the program. Their responses pointed to the benefits of continuous teacher training versus short-term training. Compared to many other educational policies and systems that support teachers in their professional development, this type of training is not sufficiently available in Croatia. Based on the reflections of the LLP “Enactive Learning in Mathematics”, in this paper we argue for a stronger presence of continuous professional development (CPD) for mathematics teachers.

2. Lifelong Learning Program “Enactive Learning in Mathematics”

Jokić et al. (2019) found that eighth graders perceive mathematics as a useful but not very interesting subject. Enactive learning, especially the use of manipulatives, has a positive effect on students’ attitudes towards mathematics (Rukavina et al., 2010, Rukavina et al., 2012). When we do something in different contexts, we are more likely to acquire a skill and retain it to use a second time (Coles & Brown, 2013). Therefore, the use of an enactive approach in mathematics education is desirable. Consequently, mathematics teachers should be trained to successfully implement such an approach.

The Lifelong Learning Program “Enactive Learning in Mathematics” was developed based on the experience gained during the participation in the EnLeMaH project and the development of the online self-study course for teachers “Applying the EnLeMaH methodology”. The program is designed as a continuous professional development program for mathematics teachers, the expected duration is 60 hours over four months, and is classified as a “training program with ECTS credits” which earns its participants 5 ECTS credits upon successful completion. As such, it also represents a step forward in the field of non-formal education of future mathematics teachers, as it can later be included as optional content in formal teacher education. The program enables participants to develop knowledge and skills to implement an enactive teaching and learning methodology adapted to the context of digital literacy that helps make mathematics more attractive to students, and guide students in the creation of enactive materials that support their learning with household objects. It is particularly useful for teachers who want to adapt the content of their lessons to the needs of their students, incorporate enactive learning, and thus make their lessons more productive. In addition, the program enables the acquisition and/or further development of existing digital literacy skills with regard to the application of enactive learning methods in the mathematics classroom, as well as in mathematics teaching in general. The usefulness of such knowledge became clear during the recent coronavirus pandemic, and the need of mathematics teachers for such content was also evidenced by the fact that all available places were taken at the one-day in-service training event for mathematics teachers on "Enactive Learning in Mathematics" organised by the Faculty of Mathematics of the University of Rijeka in cooperation with the Croatian Education and Teacher Training Agency in early September 2022.

2.1. Erasmus+ project "Enactive Learning in Mathematics at Home"

The recent coronavirus pandemic has presented mathematics educators with new challenges in applying the enactive approach in distance education and general lockdown conditions. The "Enactive Learning in Mathematics at Home" (EnLeMaH) project focused on building pedagogical competencies among mathematics teachers to develop an enactive methodology for distance or home-based education. Implementing an enactive approach to learning mathematics in a digital learning environment is based on two assumptions: Teachers have the pedagogical skills and knowledge to apply enactive methods, and the materials for enactive work in a distance or home-based instruction are available or can be obtained without much effort. The latter often requires significant redesign of activities that teachers have been doing with students in their classes or the development of entirely new activities.

The two-year EnLeMaH project (April 2021 – March 2023) aimed to promote the acquisition of innovative digital pedagogical skills for mathematics teachers in the field of functional relationships. The topic of functions was chosen for this project because the concept of function is one of the fundamental mathematical concepts and as such is a very important topic in school mathematics. However, there are not many materials for enactive activities with functions. Jukić Matić et al. (2022) emphasized the importance of understanding the concept of function and its various representations and pointed out the correlation between the approach to teaching the concept of function and the ability to apply it. Enactive learning with prominent topics in the field of functions would enable students to deepen their understanding of the concept of function and distinguish its different representations, thus avoiding formalisms and common errors made by students.

The EnLeMaH project achieved its goal through a set of specific objectives as follows (EnLeMaH, 2023):

O1 – Guidelines for enactive mathematical learning at home (EnLeMaH Guidelines, 2023),

O2 – Online teacher training course "Applying the EnLeMaH methodology",

O3 – EnLeMaH repository of enactive learning materials.

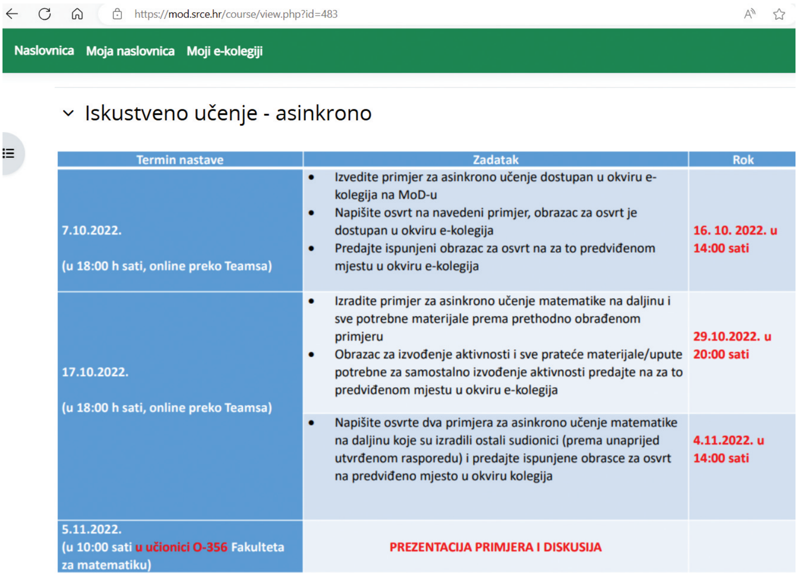
The main outcome of the project is a freely available online teacher training course "Applying the EnLeMaH methodology" designed for self-study by interested teachers (EnLeMaH, 2023).

2.2. Implementation of the Program

The Lifelong Learning Program "Enactive Learning in Mathematics" was held for the first time in the winter semester of the academic year 2022/2023. As planned, the program lasted about four months, from the beginning of October 2022 to the second half of January 2023. In other words, the program was conducted during the entire semester. The participants in the program were mathematics teachers who expressed interest after the program was announced at the one-day professional development session for mathematics teachers on enactive learning in mathematics in early September 2022. Of the 50 teachers, 14 expressed interest in participating in a continuous professional development program on the same topic. The final list of participants included nine teachers, four from elementary schools and five from middle schools (two from high schools and three from vocational schools). All participants participated in all program activities and completed all assigned tasks. Thus, all of them successfully completed the program and achieved 5 ECTS credits.

If teachers are to empathize with their students, they should have the same experiences as their students (Abrahamson et al., 2020). Therefore, this LLP was partially organized as an online course that promotes an enactive approach in a digital environment. On the other hand, active participation in the analysis of examples presented by the instructors and also by the other participants is an important part of the program. For this reason, part of the LLP has been organized in the classroom to allow unhindered exchange of views and simultaneous communication of several participants, if needed. In addition, participants in this program experienced both roles, student and teacher, in an enactive mathematical learning through distance education. During the duration of the course, participants had the following tasks: complete a self-assessment test on the Guidelines for enactive mathematical learning at home (EnLeMaH Guidelines, 2023) and Brunner's learning theory, analyse examples of synchronous and asynchronous online learning created by course instructors, design, prepare, and present an example of synchronous and asynchronous online learning, analyse examples of synchronous and asynchronous online learning created by other course participants, design of a synchronous online learning project, implementation with students in school and presentation of results, analyse projects created by other course participants. While the tasks prepared and presented by the lecturers were from the field of functions, the participants were free to choose the mathematical content for their tasks.

It is not our intention to describe the mathematical content of this programme in detail, as this is not the goal of this paper. Moreover, the course "Applying the EnLeMaH methodology" (EnLeMaH, 2023) is freely available and most of the content can be found there. However, to demonstrate the organization of the course and how the on-site work is combined with the online activities in this LLP, we reproduce part of the LLP schedule for the module "Asynchrounuos Online Learning" (Figure 1), which was available to the participants through the e-learning platform Moodle (MOD, 2023).


https://mod.srce.hr/course/view.php?id=483

Naslovnica Moja naslovnica Moji e-kolegiji

Iskustveno učenje - asinkrono

Termin nastave	Zadatak	Rok
7.10.2022. (u 18:00 h sati, online preko Teamsa)	• Izvedite primjer za asinkrono učenje dostupan u okviru e-kolegija na MoD-u • Napišite osvrt na navedeni primjer, obrazac za osvrt je dostupan u okviru e-kolegija • Predajte ispunjeni obrazac za osvrt na za to predviđenom mjestu u okviru e-kolegija	16. 10. 2022. u 14:00 sati
17.10.2022. (u 18:00 h sati, online preko Teamsa)	• Izradite primjer za asinkrono učenje matematike na daljinu i sve potrebne materijale prema prethodno obrađenom primjeru • Obrazac za izvođenje aktivnosti i sve prateće materijale/upute potrebne za samostalno izvođenje aktivnosti predajte na za to predviđenom mjestu u okviru e-kolegija	29.10.2022. u 20:00 sati
	• Napišite osvrte dva primjera za asinkrono učenje matematike na daljinu koje su izradili ostali sudionici (prema unaprijed utvrđenom rasporedu) i predajte ispunjene obrasce za osvrt na predviđeno mjesto u okviru kolegija	4.11.2022. u 14:00 sati
5.11.2022. (u 10:00 sati u učionici O-356 Fakulteta za matematiku)	PREZENTACIJA PRIMJERA I DISKUSIJA	

Figure 1. The schedule for the module “Enactive Learning of Mathematics – Asynchronous Online Learning”.

The translation of the content shown in Figure 1 is given in Table 1.

Table 1. LLP schedule for the module “Asynchrounuos Online Learning”.

Class term	Task	Deadline
07/10/2022 (at 18.00, online via Teams)	• Perform an example of asynchronous learning available within the e-course on the MoD • write an evaluation of the above example, the evaluation form is available in the e-course • submit a completed evaluation form in the e-course	16/10/2022 at 14.00
17/10/2022 (at 18.00, online via Teams)	• create an example of asynchronous distance learning and all required materials according to the previously worked example • submit the activity completion form and any supporting materials/instructions required to complete the activities independently to the location specified in the e-course	29/10/2022 at 20.00
	• write reviews of two examples of asynchronous distance learning math activities created by other participants (according to a predetermined schedule) and submit them to the location specified in the e-course	04/11/2022 at 14.00
05/11/2022 (at 10.00 a.m., Faculty of Mathematics, Room O-356)	**Presentation of examples and discussions**	

In this module, the instructor prepared an example "Activity with water" aimed at introducing a linear function in the seventh grade of elementary school through enactive learning in the form of an asynchronous online activity. The material consists of a sheet with a detailed description of the task (for the teacher), a worksheet with instructions (for the students) and solutions (for the students). The student worksheet includes a detailed explanation of the materials needed and how they can be substituted if something is not available (Figure 2), as well as tasks for enactive learning at home (Figure 3).

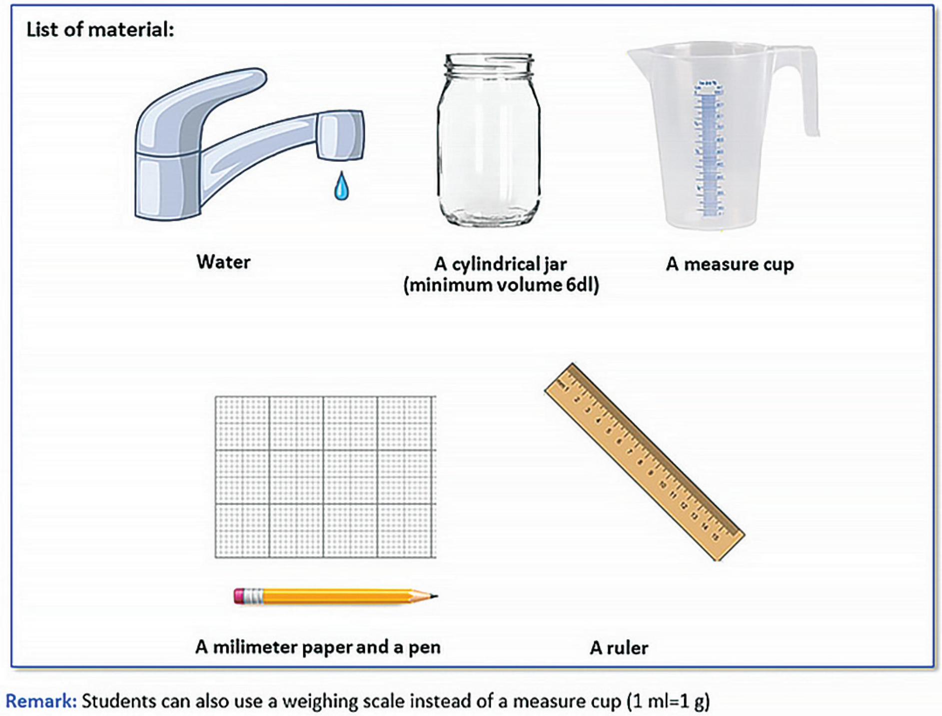


Figure 2. Material that the students should prepare for the "Activity with water".

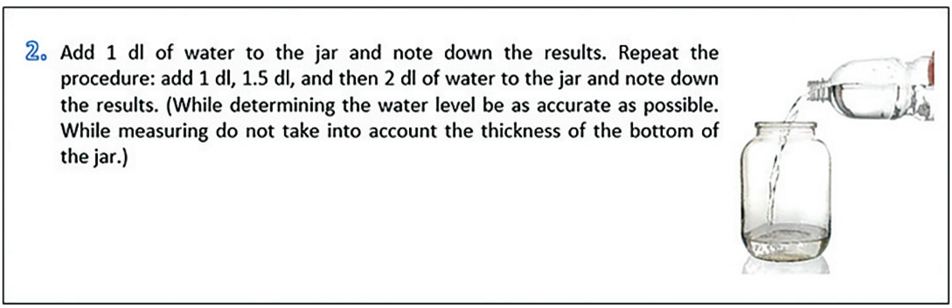


Figure 3. One of the enactive tasks in "Activity with water".

Participants were asked to perform the instructor-prepared "Activity with water" task in the role of a student and then reflect on the activity. Following the collaborative discussion of this activity, which took place in an online environment, sheets for asynchronous enactive learning at home were provided by the instructors for two additional examples, "A circle and coins" (STEM learning, 2023) and "Paper folding" (folding a paper into the shape of a square at a ratio of 1:2 – applying a linear function). Participants were then asked to create their own example and work through and reflect in writing on the examples provided by two of their classmates. The module on asynchronous enactive learning at home ended with an on-site meeting in a classroom where participants presented their activities and discussed their comments with colleagues. This was to be used to improve the activities presented when some of the colleagues commented that the materials were not clear enough. The participants said that it was very helpful to take on the role of a student to understand how detailed the materials needed to be prepared.

Similarly, enactive learning of mathematics was introduced and taught for a synchronous online environment. In this module, the introductory synchronous online activity "Functions with LEGO" prepared and conducted by the instructor deals with the first steps in the field of functions conducted in Croatia in the second grade of middle school (Figure 4).

Teacher: For our activity I prepared two sets of Lego cubes – one set consisting of 8 yellow Lego cubes and the other set which have 7 red Lego cubes as elements. Please describe your materials (discussion follows).

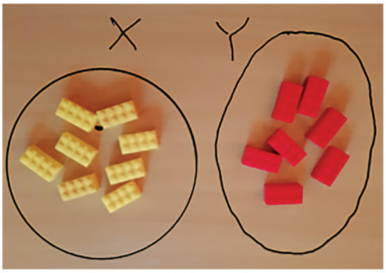


Figure 4. Part of the material for teachers in "Functions with LEGO" activity.

To put participants in the role of students, this activity was conducted as a synchronous online activity before participants had insight into the prepared materials. The following steps were similar to those of the module on enactive learning of mathematics in asynchronous online activities, taking into account the need for online presentation of synchronous online activities.

Once participants had mastered the basic stages of preparing and delivering active learning in a digital environment, they prepared a project consisting of preparing enactive tasks to deliver as synchronous online lessons with their students as part of their regular teaching. At a final on-site meeting, they reported on the activities conducted with their students and the other participants reflected on them and on the prepared material.

In total, each participant created at least three teaching materials for enactive learning in distance education in mathematics. They were free to choose the mathematical content, and an overview of the mathematical content they chose is shown in Table 2.

Table 2. An overview of the mathematical content of examples created by the LLP participants.

Asynchronous enactive learning tasks	Synchronous enactive learning tasks	Projects
• The length of a circle • Axial symmetric figures • The area of a triangle using radius of its circumscribed circle • Central and inscribed angles, Thales' theorem • Linear functions • The graph of a linear function • Inversely proportional quantities • Exponential functions • Arithmetic and geometric sequences	• The representation of an irrational number on the number line • Common divisors of positive integers • The concept of fractions • Sets and set operations • Geometric series • Mathematical induction • Algebraic expressions	• Heron's formula • The radian measure of an angle • Plane isometries • The area of rectangles with equal perimeters • A segment bisector • Nets of a rectangular parallelepiped • The volume of a geometric body • Geometric series • Systems of linear equations

The frequency of certain mathematical domains in the tasks created by the participants is shown in Figure 5.

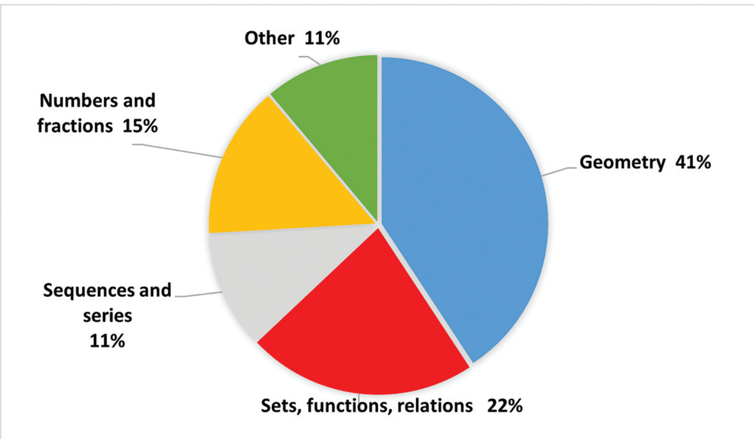


Figure 5. Frequency of certain mathematical areas in the tasks of the participants.

As might be expected, geometry accounts for most of the content, since enactive learning strategies naturally emerge when this part of mathematics is presented. This is followed by content on sets, functions, and relations. We believe that the number of examples from "sets, functions, relations" is influenced in part by the examples given by the LLP instructors. Furthermore, the ways participants created their materials and the strategies they used for enactive mathematics instruction within a selected topic were largely consistent with the strategies presented by LLP instructors.

All materials prepared during the LLP were discussed in detail, and the instructors were available for consultation during the preparation of the materials. However, it happens in many situations that participants ask each other for advice based on their teaching experience rather than asking the instructors when in doubt. We consider this networking among participants and collaboration in the preparation of projects to be another positive outcome of the LLP that can only be achieved through continuous professional development.

3. Teachers' reflections on participation in a continuous professional development program "Enactive Learning in Mathematics"

Given the duration and demands of the program, it is certain that participation in the program placed a significant additional burden on the participating teachers. However, during the implementation of the program, the satisfaction of the program participants was evident as they repeatedly expressed uncoerced positive comments. They emphasized the benefits that the experience of program participation brought to their daily work, as well as the fact that they would implement some of the examples presented in the program by instructors or other program participants. In addition, after completing the program, participants filled out an online questionnaire about the implementation of the program. The organization of the Lifelong Learning Program received the highest marks in all categories from all participants, and all comments made in the questionnaires are praise for the organization of the program.

According to OECD (2019), teachers who report participating in impactful training tend to display higher levels of self-efficacy and job satisfaction. In addition, teachers participating in training focusing on the implementation of pedagogical practices tend to report a more frequent implementation of effective practices. To improve the performance of school mathematics, we need to better understand the underlying characteristics of mathematics teacher education and the professional development contexts that have a positive impact on teachers' professional learning (Zehetmeier et al., 2020).

Since all participants of our program expressed a high motivation to continue participating in similar programs, we wanted to find out what elements of the program motivated them to continue participating in continuous professional development programs. For that purpose, we conducted individual interviews with program participants, the results of which are presented below.

3.1. Interviews with the participants

In the group discussions held during the program, all participants were extremely positive about the activities carried out and fulfilled the tasks assigned to them with great enthusiasm. Individual interviews were conducted with the participants after the completion of the program. They were encouraged to express their opinions about the EnLeMah program and highlight what they think should be changed and what is best about the program. We also asked them about their participation in professional development in general: how often they participate in professional development, what their experiences are, and how they are supported by their schools. Finally, we asked them to compare this program to other forms of professional development they have participated in, and to explain what their motivation is to participate in continuous professional development programs in the future. They also indicated the topics in which they were most interested. Below is an overview of the information collected.

Participants

Nine teachers participated in the program, four from elementary schools and five from middle schools (two from high schools and three from vocational schools). All participants were female. The average professional school experience of the participants was 14 years (one teacher with one year of experience, one teacher with 31 years of experience, and all other participants had between 8 and 19 years of professional school experience). All participants regularly participate in professional development activities.

Previous professional development (and school support)

All participants regularly attend local training events (at the school, city or county level). The duration of these events is usually several hours. Program participants also regularly attend online training at all levels. They are interested in participating in multi-day training programs at the state level, but this is highly dependent on financial support from the school and the organization of teaching at the school. So the principal plays an important role. In general, they give support, but the financial situation of the school is the problem.

For example, one participant stated that she regularly attended state-level trainings when they were held nearby. Now that these trainings are held in a more distant location, she no longer goes every year because the school has no funding and only one math teacher from the school can attend the annual state meeting. Similarly, the second participant states that in the nearly 10 years she has worked at the school, she has only attended in-service training at the state level twice (not including online training). The other participants also had such experiences. One of them explained that each year one or two mathematics teachers from their school are selected to participate in the state event, with preference given to those with longer experience and higher rank. Thus, participation in the state-level events is not evenly distributed among teachers.

Regarding continuous professional development programs, participants took part in them as part of the recent "School for Life" curriculum reform in Croatia or when they participated in other projects (e.g., the "E-Schools" project) that required participation in such programs. They are not aware of any other such programs, nor have they participated in any such programs that were not part of a project.

Reflections on participation in the LLP "Enactive learning in Mathematics"

During the interviews, we did not receive a single negative comment about the content or organization of the program. All participants highlighted the good organization, the high applicability of the content, and the opportunity to discuss and share experiences with other participants as positive aspects of the program.

We provide an overview of feedback we received about the program.

- In the first activity presented, everything was explained in detail, detailed instructions were given so we knew what to do and how to do it. Positive aspects of the program: commenting on others, coming up with an idea independently - creativity (other trainings did not offer this challenge, this program raised the bar), interaction between participants in creating the tasks. (a teacher with 8 years of experience)
- It was great. It's hard when you have to come up with a new idea – but in the process you come up with many other ideas. (a teacher with 19 years of experience)
- The biggest problem is the idea, later it is not a problem anymore. At the end, the results are visible – many ideas that can be used. I got much more than I invested – I got a lot. (a teacher with 19 years of experience)
- The program provided new ideas. Positive aspects of the program: There was enough time for everything; it was well structured; others' opinions are heard. (a teacher with 1 year experience)
- It was a great, very enjoyable experience. We learned a lot. The assignments were clear, you knew what you had to do. I liked the group. I did not feel the feedback was a bad thing, it was a help on how to improve, how to do something better. I learned a lot from the others and will try to implement their ideas. (a teacher with 17 years of experience)
- I have to admit that sometimes it was very exhausting. When we were given a task, I would run out of ideas at first, but every time after thinking about it for a day or two, I would come up with something that forced me to take a different approach in class, not to mention how many good ideas we got from all the other colleagues in this training. (a teacher with 13 years of experience)
- After 31 years of service, after many reforms, after many professional councils, after the Covid, I felt good and useful. The only thing I lacked was time. Everything that was done was concrete, and that is always missing, and you

> always learn from it. Something that you can fully apply in the classroom. Great experience. It was great to work with young people and learn from them, because I still enjoy that. And of course, the assignments were short and clear. (a teacher with 31 years of experience)

Motivation for future participation in continuing education programs (and topics of interest)

As one of the program participants said, the motivation for attending professional development programs is generally to see something new that can be applied in the classroom, how to make a difference, for a different way of teaching. To get feedback from other colleagues and hear their experiences (a teacher with 17 years of experience). A similar attitude was expressed by all participants.

The topics that interest them are broad. One of the participants said, "In general, I am interested in all topics that are closely related to teaching and can help me in my daily work. Perhaps I should point out that I am particularly interested in assessment because it is one of the most difficult parts of our work" (a teacher with 13 years of experience). Other topics mentioned by participants included problem solving, working with gifted students, and working with struggling students. As methods for professional development programs, they preferred workshops, sharing ideas with colleagues, and activities outside the classroom.

4. Discussion and conclusion

As observed by the course instructors and expressed in the interviews, the participants in the LLP "Enactive Learning in Mathematics" are a group of highly motivated mathematics teachers who regularly attend professional development programs and are involved in many projects with their students. As such, they may not be typical representatives of all mathematics teachers, but they certainly have a good insight into the professional development opportunities for mathematics teachers in Croatia and a wealth of experience that allows them to identify the desirable features of such professional development, considering how much it contributes to their development as teachers. They emphasize the advantages of online professional development, as it can be attended by a large number of participants. However, they are bothered by short trainings where they have to passively listen to others' lectures. Although there are many useful lectures in online continuing education, participants expressed the opinion that the proportion of such trainings should be reduced in favor of workshops and other forms of work that require participant activity. As one of the participants said, she would rather attend a training where she has to do something rather than just listen (a teacher with 10 years of experience). Another participant said that she was attracted to LLP trainings where the work was more intense and the participants were more involved (a teacher with 19 years of experience). This is consistent with research findings that active learning and collaboration on the one hand and sustained length, on the other are desirable features of professional development programs, along with a content focus and programs embedded in schools (OECD, 2019). First, because they provide opportunities to apply new ideas and knowledge in their own classrooms, and second, because they provide follow-up activities and take place over a longer period of time.

Given recent changes in teacher promotion and rank retention requirements (MZO, 2019), it is expected that more mathematics teachers will express a desire to participate in professional development programs than in the past. For example, a minimum of 150 professional development hours in the last 5 years is required to become a teacher advisor. Given the testimony of mathematics teachers who have participated in our lifelong learning program, it is not difficult to imagine that many of mathematics teachers in Croatia would prefer to complete some of these hours in continuous professional development programs rather than accumulating the hours in one-day programs, notwithstanding the fact that longer programs impose more obligations on them. Such continuing education would likely give them more satisfaction while significantly improving their knowledge, skills, and competencies. However, the financial situation of schools is already such that it is often difficult for principals to accommodate teachers' requests for professional development. If the number of such requests increases, it is likely that the inability to participate in professional development activities will increase frustration and further complicate the situation of mathematics teachers, who are already in short supply in Croatian schools. In the long run, this could have a negative impact on the already existing shortage of mathematics teachers and minimize the impact of the recent scholarship measure for future mathematics teachers. Therefore, this is an issue that should be seriously considered in the context of education policy.

Policies that require mandatory participation of teachers in continuous professional development may reflect a particular system's efforts to ensure that every member of its workforce has access to these opportunities (OECD, 2019). Many education systems place a high priority on improving teachers' access to continuous professional development: Expanding professional development opportunities, ensuring time and leave entitlements for participation, and linking professional development to career advancement (OECD, 2020). Due to recent changes in the education system, Croatia also belongs to this group of countries. However, training in the form of one-time or short learning series provided externally is not ideal (OECD, 2020), and further efforts should be made. Access to continuous professional development should be open to all teachers equally, not only in systems where it is mandatory, but also where it is a prerequisite for career advancement. If schools do not receive adequate funding for it, access to continuous professional development for those teachers who want it must be regulated by education policy at the state level. Expanding the recently introduced adult education voucher system (HZZ, 2023) to provide continuous professional development for teachers already in schools could be one way to address the problem.

To make the implementation of CPD more effective for mathematics teachers, we need to better understand the underlying characteristics of mathematics teacher education and the professional development contexts that positively impact teachers' professional learning (Zehetmeier et al., 2020). Intrinsic factors motivation has a major impact on the implementation of CPD. The most important intrinsic factor that motivates mathematics teachers to implement CPD is increasing teachers' mathematical knowledge and improving their personal teaching quality (Anitasari & Retnawati, 2018). According to the interviews conducted, the high applicability of the content in the mathematics classroom is one of the greatest strengths of the EnLeMaH program. In addition, all participants appreciated the detailed instructions on the tasks and the clear structure of the program, and emphasized the opportunity to exchange ideas and receive collegial advice from other mathematics teachers in a friendly and collegial environment. None of these features, which participants identified as key strengths of the program, could not be achieved to such an extent through a short teacher training.

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Kontinuirani profesionalni razvoj nastavnika matematike – Refleksije na Program cjeloživotnog obrazovanja "Iskustveno učenje matematike"

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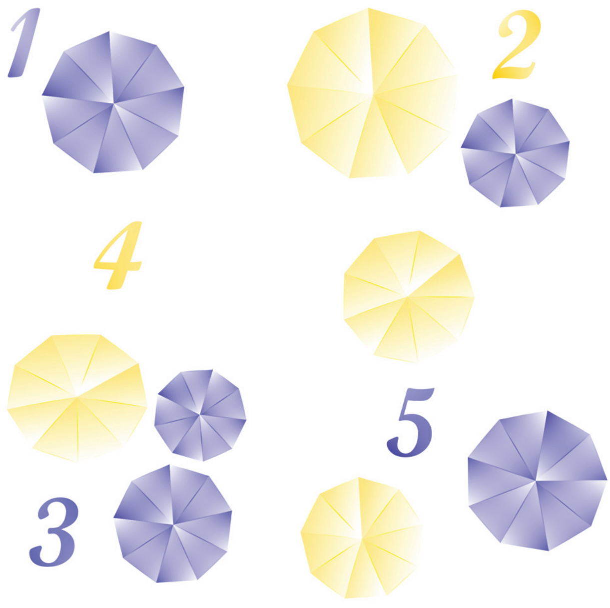
Sažetak. Program cjeloživotnog obrazovanja "Iskustveno učenje matematike" po prvi je puta izveden na Fakultetu za matematiku Sveučilišta u Rijeci u zimskom semestru 2022./2023. akademske godine. Program omogućuje polaznicima razvoj znanja i vještina za implementaciju metoda poučavanja za iskustveno učenje matematike, prilagođenih kontekstu digitalne pismenosti koja pomaže da matematika bude privlačnija učenicima, i usmjeravanju učenika u korištenju kućanskih predmeta za izradu materijala koji podržavaju njihovo iskustveno učenje. Osim toga, sudionici ovog programa doživjeli su obje uloge, učenika i učitelja, u enaktivnom matematičkom učenju. Po završetku programa provedeni su razgovori s nastavnicima matematike koji su u njemu sudjelovali. Naglasili su prednosti koje je iskustvo sudjelovanja u programu donijelo njihovom svakodnevnom radu, kao i činjenicu da će neke od primjera koje su u programu predstavili instruktori ili drugi sudionici programa koristiti u svojoj nastavi. Osim visoke primjenjivosti sadržaja u učionici matematike, detaljne upute o zadacima, jasna struktura programa, mogućnost razmjene ideja i kolegijalni savjeti drugih nastavnika matematike u prijateljskom i kolegijalnom okruženju prepoznati su kao glavne prednosti programa, utjecaj koji je teško postići u kratkim programima stručnog usavršavanja. Također smo razgovarali s učiteljima o njihovoj motivaciji za sudjelovanje u programima cjeloživotnog učenja i podršci koju dobivaju. U radu su dane refleksije temeljene na tim razgovorima i izvedenom programu. Na temelju promišljanja Programa cjeloživotnog učenja "Iskustveno učenje matematike" zalažemo se za snažniju prisutnost kontinuiranog stručnog usavršavanja nastavnika matematike u Hrvatskoj.

Ključne riječi: profesionalni razvoj, nastavnik matematike, program cjeloživotnog obrazovanja, iskustveno učenje matematike, obrazovna politika

5.
Digital Learning Environments and Their Impact on Mathematics Education



(Digital) Game-Based Learning in Mathematics

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Abstract. Mathematics is often viewed as an abstract body of knowledge that is disconnected from the real world, unlike other STEM disciplines. Thus, students may experience a sense of disconnection from the mathematical topics. Additionally, traditional teaching methods are criticized for their inability to engage students and for denying them the autonomy to construct their own comprehension. Due to its interactive nature, Game-Based Learning (GBL), particularly Digital Game-Based Learning (DGBL), appears to be a promising approach to learning and teaching of mathematics. Digital games have unique characteristics (such as fantasy, rules/goals, sensory stimuli, challenge, mystery, and control) that can increase learning motivation and result in attitude and behavior change. In contemporary education, DGBL is an effective medium for students to learn mathematical concepts and practice mathematical skills. However, implementing DGBL in teaching presents a considerable challenge for teachers. In this teaching approach, teachers are designers, facilitators, or guides. Therefore, they must be assisted in acquiring the knowledge and skills necessary to assume these roles.

Keywords: DGBL, digital games, mathematics teachers, GAMMA project

1. Introduction

Despite the importance of mathematics, the majority of elementary and secondary students view it negatively and perceive it as a frustrating and difficult subject that causes learning fatigue, pressure, and anxiety (e.g., Deng et al., 2020; Sun et al., 2021). Furthermore, mathematics is the school subject with the highest rate of student failure (e.g., Hung et al., 2014). Digital game-based learning (DGBL) is a promising method for overcoming students' frustration and anxiety, boosting

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motivation to learn mathematics, and advancing mathematical knowledge. DGBL incorporates educational goals and educational content into games, fostering a fun and engaging learning environment that encourages students to learn and advance their knowledge and skills (Prensky, 2001; Hussein et al., 2022). Digital games allow players to learn by doing and help them develop transferable knowledge and skills for the real world (An & Bonk, 2009; Hwa, 2018). In this paper, we provide an overview of the current state of DGBL in mathematics and describe the outcomes of the Erasmus+ project GAMMA (GAMe-based learning in mathematics).

2. Game-Based Learning

Each year, teachers encounter new ideas, methods or teaching strategies as if they were the answer to every issue in mathematics education. However, a single solution will not completely improve teaching and learning. The use of game-based learning (GBL) in the classroom is one of these innovative methods or approaches. GBL is a method of learning that employs the concept of games to achieve specific knowledge, skills, or attitudes. It is frequently defined as encompassing all aspects of using games to teach and learn. Therefore, people often confuse GBL with gamification, a term that describes the application of game design elements in non-gaming contexts. GBL engages and supports students in productive struggle by utilizing a problem-based learning approach, learning through failure, insightful feedback on learning, and the experience of progressive growth (Schrier, 2018). All these aspects can be linked to effective mathematics teaching practices (NCTM, 2014). Digital games, playable on a variety of electronic devices such as smartphones, may hold greater relevance for 21st-century students compared to traditional board games. GBL becomes digital game-based learning (DGBL) when digital games are used (Prensky, 2001; Deterding et al., 2011). DGBL integrates digital educational games and self-directed learning into teaching, enabling learners to participate in immersive learning experiences while acquiring knowledge and skills.

Several terms have been used interchangeably in the context of DGBL: serious games, educational games, and digital educational games. These three concepts have commonalities but also distinctions. Serious games are interactive games that allow players to carry out activities that enable them to practice skills and achieve aspects beyond simply enjoying a leisure activity (Pan et al., 2021). They use the same medium as video games intended for recreational play. The content of serious games includes personnel training, policy discussion, military training, education, health, medical treatment, and so on. Educational games are those specifically created for educational purposes (e.g., Ahmad et al., 2015). They can be both physical and digital games (Vos et al., 2011). Digital educational games (sometimes referred to as educational video games) are educational games in digital form (Hussein et al., 2022). Digital educational games require information technology equipment and various digital platforms to support game development. Such games must also include educational features that enhance students' comprehension of the subject matter. In this paper, we will use the term digital educational games (or digital games).

3. Effectiveness of DGBL in Mathematics Classroom

Digital games have been developed and implemented primarily as supplementary educational tools. Nonetheless, the current state of technology enables the use of games as a main instructional tool. Some games, for example, can adjust to students' varying abilities and provide teachers with progress reports to assess students' understanding of the subject matter and provide feedback on any areas where they require additional assistance (Callaghan et al., 2018). DGBL is also linked to critical thinking, creativity, problem-solving, agency, collaboration, communication, and digital literacy, which are all important 21st-century skills (Gee, 2005; Williams-Pierce, 2019). Most math educators agree that, when compared to other subjects, mathematics necessitates a unique set of skills for teaching and learning. Thus, games designed and used for math education can be distinct from those for other subjects (Pan et al. 2022). This calls into question the efficacy of DGBL in mathematics education.

Pan et al. (2022) conducted a systematic review to determine how and what kinds of digital games are used in mathematics education. The review included 43 studies, 40 of which used experimental design, while two used non-experimental designs (qualitative inquiry and design-based research). Review detected that studies taking place in the regular school classes seemed relatively more effective than those performed in after-school programs as 3 out of 31 and 1 out of 7 respectively reported negative findings. The majority of studies reviewed found that educational standards, such as state curricula, state assessment standards, and national council standards, were aligned with the learning content in math games. Game play was most effective for improving students' learning behaviors, followed by working memory, then knowledge and skill acquisition. Furthermore, the review discovered that digital math games are frequently created and used as supplements in traditional classrooms; seldom games have been used to develop new knowledge. The majority of the games were drill-and-practice exercises. To respond to questions in games, players were expected to use prior knowledge and apply it, reinforcing their knowledge application or engaging in drill-and-practice through gameplay. The most popular game mechanics was time-limited in-game competition with either computers or other players, especially when the learning objective was to improve students' computational fluency. The most popular games for this type of achievement were action, puzzle, and strategy games. Role-playing games, adventure games, simulation games, and construction games were frequently used to engage students when the learning objective was to achieve higher-order cognitive learning outcomes. These games frequently supported the experiential learning approach in game design.

Hussein et al. (2022) conducted a more rigorous systematic literature review, analyzing only studies with quasi-experimental and randomized control trial designs. Participants in the majority of studies were elementary school students. The findings revealed that GBL was used to teach a variety of mathematics-related topics, with an emphasis on arithmetic operations. Review also revealed that 22 of 27 studies in the category of knowledge acquisition yielded positive results, while other studies yielded mixed findings. Seven of the nine studies found that perceptual and cognitive abilities improved, one study reported neutral outcomes and one study reported negative outcomes. Six of the seven studies on affective, motivational, and behavioral change were successful, while one study reported neutral outcomes. Considering all of the findings, one can conclude that DGBL has a promising future in mathematics education, particularly at the elementary level and for knowledge acquisition.

Drill-and-practice games were utilized in the majority of DGBL studies on mathematics education, according to these two most recent reviews and one earlier by Byun and Joung (2018). Moreover, those reviews show there are few on DGBL with high school mathematics compared to middle or primary education. On one hand, it is possible high school mathematics is more difficult to implement in digital educational games. Simpler mathematical concepts and procedures from primary and middle school can be easily transformed into drill and practice games. Kafai & Burke (2015) refer to this issue as a deeper philosophical matter hidden within the premise of the educational games: the need to "sweeten" the learning of difficult or boring ideas through games. On the other hand, such studies may be lacking due to challenges mathematics teachers encounter in implementing DGBL.

4. Pedagogical Issues Regarding DGBL

Mathematics teachers' beliefs influence their classroom practices as well as their perceptions of teaching, learning, and assessment (e.g., Barkatsas & Malone, 2005; Liljedahl, 2008). What makes a school a DGBL environment is closely related to how teachers think about games. This encompasses their perspectives on the effectiveness of games, their limitations, and the appropriate and inappropriate use of games (Beavis et al., 2014). The manner in which teachers use games has a direct impact on the success or failure of DGBL initiatives (Mehrotra et al., 2012). Huizenga et al. (2017) discovered that teachers' intention to teach with digital games is positively related to the perceived usefulness of digital games, such as helping students develop mathematical skills (Yeo et al., 2022).

New technological advancements have sparked the development of educational games designed to enhance learning and integrate with teachers' practices and curricula (Callaghan et al., 2018). Thus, teachers now face new difficulties in integrating these tools into their curricula for the benefit of their students. This explains the gap between teachers' desire and practice of using games for teaching and learning (Shah & Foster, 2015). Related research has found that teachers, even if they find digital games fun or potentially useful for learning, either avoid using them as a teaching tool or use rather simple forms of games in their classes, such as drill-and-practice or trivia games (Jesmin & Lay, 2020; Takeuchi & Vaala, 2014). Teachers express frustration at finding appropriate games that align with their curriculum and incorporating games into their lessons (Takeuchi & Vaala, 2014). They also link a lack of funding to the inability to purchase high-quality educational games and insufficient technical equipment in schools, which affects the ability to implement DGBL (Baek, 2008; Kaimara et al., 2021; Li, 2017). In terms of support, teachers state that the lack of an educational policy, as well as a lack of professional development opportunities, impedes the adoption of digital games and DGBL in the classroom (Kaimara et al., 2021).

Teacher DGBL competencies and game-based pedagogy research are still ongoing, and little is known about how to best support teachers in integrating educational games into classroom instruction (Callaghan et al., 2018; Takeuchi & Vaala, 2014). Nousiainen et al. (2018) identified four key DGBL competencies for teachers: collaborative (teachers' ability and willingness to share and communicate content, ideas, and practices), technological (overcoming technological barriers and analyzing technological tools), pedagogical (curriculum-based planning, in-game tutoring, and assessment), and creative (ability to take a playful stance, explore, and improvise). Kangas et al. (2017) defined game-based pedagogy as activities such as planning, orientation, playing, and elaboration. The final step, elaboration, is critical for learning because it allows instructors and students to discuss the game, clarify points, and reflect on what they have learned (Bado, 2022).

5. Educational Effectiveness of Digital Games

Digital games as a whole can be difficult to study because there are numerous game types with radically different core mechanics, making it difficult to consider *digital educational games* as a unified instructional approach. Recently, researchers have made greater efforts to create educational games with the content and outcomes desired by teachers (Callaghan et al., 2018). Such games are not commercially available for a broad audience, unlike commercial games labeled as educational. However, there is no verification that commercial games enhance learning. A recent study examined commercial apps labeled as educational games for mathematics and digital worksheets with immediate feedback and interactive elements for the same mathematical content (Gresalfi et al., 2018). The researchers found that the game-like apps supported similar transfers of knowledge as the worksheet-like apps. However, students found that games were more enjoyable and engaging than the worksheets.

When evaluating the educational impact of digital games, it's crucial to focus on the quality and nature of the learning experiences they facilitate, rather than just their built-in features (Pan & Ke, 2023). DGBL enhanced with specific learning supports offers a strategic approach to augmenting educational outcomes, indicating a shift towards a more tailored and effective use of educational technologies. Incorporating learning supports within DGBL has been demonstrated to deepen learners' engagement and understanding significantly:

1. Knowledge transformation: Students are able to convert tacit, implicit knowledge into explicit knowledge that they can consciously access and communicate (ter Vrugte & de Jong, 2017).
2. Learning Transferability: These supports help learners apply the skills and insights gained from gameplay in external assessment contexts, thereby validating the practical utility of game-based learning (Bainbridge et al., 2022).
3. Cognitive Processing: There is a shift from intuitive to generative processing, enabling students to create new knowledge and solutions independently (O'Neil et al., 2014).

4. Problem Solving: Learners learn to externalize problem representations, which can improve their ability to solve complex problems by making the problem-solving process more visible and explicit (Lee & Ke, 2019).

However, the integration of learning supports within games is not without challenges. These enhancements can sometimes disrupt the flow of the game, which might affect the engagement and immersive experience that are critical to the effectiveness of game-based learning (Pan & Ke, 2023). This complexity underscores the need for careful design and implementation of learning supports to balance educational goals with engaging gameplay. Thus, the pursuit of enhancing DGBL with additional features is a nuanced endeavor that requires thoughtful consideration of both educational and game design principles. While research has affirmed the educational potential of digital games, it has not yet established a comprehensive set of core design principles for creating effective educational games (Clark et al., 2015; Gresalfi et al., 2018). Developing such principles would not only guide future game designers but also enable researchers and educators to evaluate and compare the effectiveness of various games in fostering specific learning outcomes (Gresalfi et al., 2018; Larkin, 2015).

5.1. Examples of Digital Math Games

Table 1. Digital math games.

Name	Level	Topic	Link
DragonBox (Algebra 12+) and its newer version: Kahoot! Algebra 2 by DragonBox	For ages 9 and older	Algebra	`https://dragonbox.com/products/algebra-12`
DragonBox (Elements) and its newer version Kahoot! Geometry by DragonBox	For ages 9 and older	Geometric shapes	`https://dragonbox.com/products/elements`
Faktr – all access	For ages 12 and older	Multiplication and division (factors)	`https://www.taptap.io/app/60591`
Jabara	Secondary school	Algebraic simplification	`https://www.mangahigh.com/en/games/jabara`
Ratio rumble	Middle school	Ratios	`https://mathsnacks.com/ratio-rumble.html`
Wrecks Factor	Secondary school	Factorizing quadratics	`https://www.mangahigh.com/en/games/wrecksfactor`
Variant: Limits	High school	Function limits, asymptote, continuity	`https://triseum.com/variant-limits/`

Some of the digital math game platforms are Math Snacks (`https://mathsnacks.com/`), Math Playground (`https://www.mathplayground.com/`), Legends of Learning (https://www.legendsoflearning.com/). These platforms offer games that cater to the mathematical content of elementary school, providing opportunities for learning and practice. In the table above (Table 1), we list some of digital games. List of platforms and mathematical games can be seen at http://www.project-gamma.eu/.

6. GAMMA – GAMe-based learning in MAthematics

From the literature review, the following issues emerged: a) there are not many digital games for high school mathematics; b) existing games focus mostly on drill-and-practice; c) the teachers do not have clear guidance or knowledge on how to implement digital games into mathematics lessons. The mentioned issues encouraged us to start a project called GAMMA – GAMe-based learning in MAthematics. Within the project, we created a handbook for teachers on DGBL, focused authoring systems (FAS) for game development, digital games for learning high school mathematics and teaching scenarios which support the implementation of digital games created.

6.1. Handbook for Teachers and Teaching Scenarios

The handbook describes the instructional design of the DGBL lesson, which should serve as a guide for the DGBL process in the classroom: what teachers must pay attention to when implementing games (resources, infrastructure, time), what strategies can be employed before, during and after gaming, and examples how the lesson should look like. One of the important advices for teachers is that teachers must play the game themselves before giving it to students to be able to connect the game mechanics with the mathematical content and to understand how the game dynamics and aesthetics affect the creation of mathematical knowledge. The handbook describes the current trends in DGBL in mathematics and its benefits, along with some concrete examples of empirical studies. It also includes an overview of available commercial math games for high-school mathematics on various game platforms, as well as an evaluation of some of these games based on a number of criteria. For the evaluation of games from the point of view of learning and knowledge acquisition, the factors identified and classified by Giani & Wangenheim (2016) are used: learning, social interaction, challenge, competence, immersion, fun, relevance, clarity of purpose, usability, motivation, satisfaction, feedback and curiosity. Also described are the fundamentals for educational game design, which can assist teachers in designing their own games. The teaching scenarios are created as the blueprint for DGBL lessons using GAMMA games as part of this project. In the scenarios, we highlighted crucial moments that the teacher must accomplish during the lesson. We did not create a lesson plan because we wanted to give teachers the freedom to design the entire lesson and come up with activities beyond those listed in the teaching scenario. Furthermore, teachers, who were project participants, have differing views on how structured teaching scenarios should be. Consequently, we attempted to reconcile diverse educational cultures. For instance, Finnish teachers do not desire a detailed lesson plan, whereas in the new situation, Croatian teachers prefer prescribed procedures. Each teaching scenario contains the following elements: mathematical domain and learning outcomes, keywords, description of the game, age range of the students, prerequisite knowledge for student, prerequisite knowledge for teacher, resources needed and description of activities (pre-game, in-game and after-game activities) (Jukić Bokun et al., 2022).

6.2. Design and Implementation of the GAMMA Games

Here we present the two GAMMA games as well as the result of their implementation in the classroom.

6.2.1. GAMMA ProbChallenge

ProbChallenge is a narrative single-player game. The player is an intern who reports to the laboratory for work. The player becomes acquainted with fundamental probability concepts and calculates probabilities in order to become an equal member of the laboratory (Figure 1). There are five levels in total. The player can choose which of the first four levels to play, but must achieve a certain score on the fourth level to access the fifth. The game is available free of charge in the App Store and the Google Play.

In developing the ProbChallenge game, we used the approach described by Aleven et al. (2010). The authors provide a general framework for designing and analyzing educational games. The framework is based on three main parts: setting learning objectives, using the MDA framework and incorporating instructional design principles, as well as a strategy for how to combine these components effectively so that the designer can keep the whole picture in mind. The MDA framework is a game-design framework that relates game mechanics (M), dynamics (D) and aesthetics (A) (Hunicke et al., 2004): The mechanics of a game refer to the particular components of the game, the dynamics of the game refer to the behaviours that result from the application of the game mechanics and the player's input during gameplay, and the aesthetics of a game refer to the player's subjective experience during the interaction with the game system. The inclusion of instructional design principles is based on the assumption that instructional design principles established in other types of learning environments can be applied to educational game design and can help develop games that are educationally effective. Examples of such principles are the Multimedia Principles (Mayer, 2014, 2019) and the Lifelong Learning Principles (`https://louisville.edu/ideastoaction/-/files/featured/halpern/25-principles.pdf`). There is overlap between these collections of principles, but they have a slightly different focus. A designer of educational games has to think almost constantly about how to combine components to design/redesign the game so that the components work in concert. For example, a useful way to expand the learning objectives for a given game is to look at the knowledge and skills that were originally considered as "transfer" and to consider whether they can be brought into the game. To design or redesign a game, one might think in terms of MDA to come up with ideas that improve or change the aesthetics and maybe the dynamics of the game. A final source of redesign ideas might be to consider whether additional instructional principles can be introduced into the game to increase the educational effectiveness of a game.

Setting learning objectives helps designers ensure that the game they develop actually meets a set of intended and coherent learning objectives. These include (a) a written specification of prior knowledge and skills; (b) examples of tasks by which a student/player will improve the given knowledge and skills; and (c) reflecting about the potential transfer – what knowledge and skills might they learn that go beyond what they actually encountered in the game?

During brainstorming sessions with teachers from Croatian GAMMA partner schools, we decided to use the game to achieve the following learning objectives: to describe a definite and impossible event; to use event algebra (union, intersection, complement) to calculate probability. We also agreed to create a game for learning new content. The teachers from the partner schools proposed the main idea for the story. In implementing the MDA, we considered Aleven et al.'s (2010) suggestion that a slower and more deliberate dynamic may be more appropriate if the game's learning objectives require the student or player to make sense of the embedded learning content. Given that ProbChallenge helps with learning, we used cognitive load theory by using modeling. We stayed away from animations because they negatively impact working memory. Moreover, we have combined two different kinds of information: a picture and a story (Sweller et al., 2019). When it comes to the effectiveness of modeling in the classroom, scaffolding and worked-out examples are the two most common types of modeling with supporting evidence, particularly in mathematics education (Lee & Ke, 2019). Throughout the game development process (design/redesign), we were constantly relying on instructional design principles to create the game. Overall, we have taken the following Multimedia principles into account in the development of the game:

- personalization – using conversational language rather than formal language
- coaching/feedback – adding in-game advice and feedback;

and the following implications of the Life-Long Learning principles:

- learning environments should capture the important content in stories and example cases, which are comparatively easy to comprehend and remember
- provide learners with multiple and varied examples of concepts
- plan the order and amount of new information that is to be presented in discrete units so as not to overwhelm new learners with too much new information at once
- learning environments promote cognitive flexibility by having students work on problems that vary in content and complexity
- learning environments should tailor the materials to characteristics of the learner, making sure that the material is not too difficult
- provide immediate feedback on errors

- learning environments should deliver good explanations of ideas and elicit self-explanations from the learner.

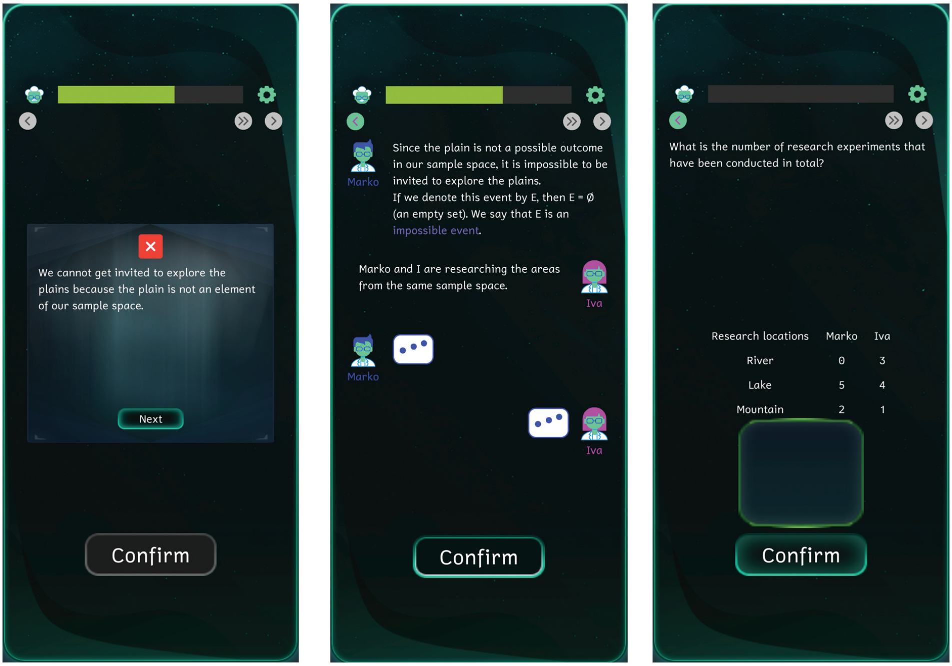

Figure 1. Screenshots from the GAMMA ProbChallenge game showing from left to right (1) the introduction of an impossible event, (2) a help message in the case of a wrong answer, and (3) a sample question related to the calculation of the probability.

Members of the GAMMA project and students from partner schools tested the beta version of the ProbChallenge game. The game was revised based on their comments. The revised version of the game included the addition of two levels to the first three levels so that the player can systemize and apply the content taught in the first three levels on the various probability tasks, more detailed feedback on incorrect answers and performance, and the addition of sound and translation of the English version of the game into the GAMMA partner languages: Croatian, Dutch, Finnish, and Greek. The following instructional design principles were integrated into the teaching scenarios: the teachers need to provide guidelines and explicit instruction for the principles that are to be learned, and according to the multimedia principle of pretraining, it is recommended to provide pre-game information regarding content and/or mechanics of the game.

6.2.2. E(qua)scape

E(qua)scape is an escape room game in which a system of linear equations should be solved with the help of simulations (Figure 2). The design of the game E(qua)scape began with the setting of learning objectives, as described by Aleven et al. (2010)

and used in the GAMMA ProbChallenge design process. The game is created using Machine Lab Turtlesphere 2 (MaLT2, `http://etl.ppp.uoa.gr/malt2`). MaLT 2 (Kynigos & Grizioti, 2018) is an online environment that combines Logo-based programming with dynamic manipulation and 3D navigation for the exploration of mathematical concepts. The games created with MaLT2 rely on constructionism. Constructionists have always prioritized developing transformative activities that allow students to create meaningful digital artifacts that represent their own understandings of mathematics (Girvan & Savage, 2019; Kynigos, 2015; Papert, 1993). According to the constructionist approach, students can have more opportunities to form new connections with knowledge by participating in the process of creating or managing a game and using mathematics as a tool, as opposed to having traditional lessons embedded directly into games (Kafai & Burke, 2015). In this perspective, the main idea is to create a microworld that simulates a real-world situation or phenomenon and provides students with functionalities to explore, change, and extend the initial rules and behaviors of the simulation. MaLT2 has a strong connection with mathematics education, but microworlds have also been used in activities for other scientific domains, such as physics, engineering, and history.

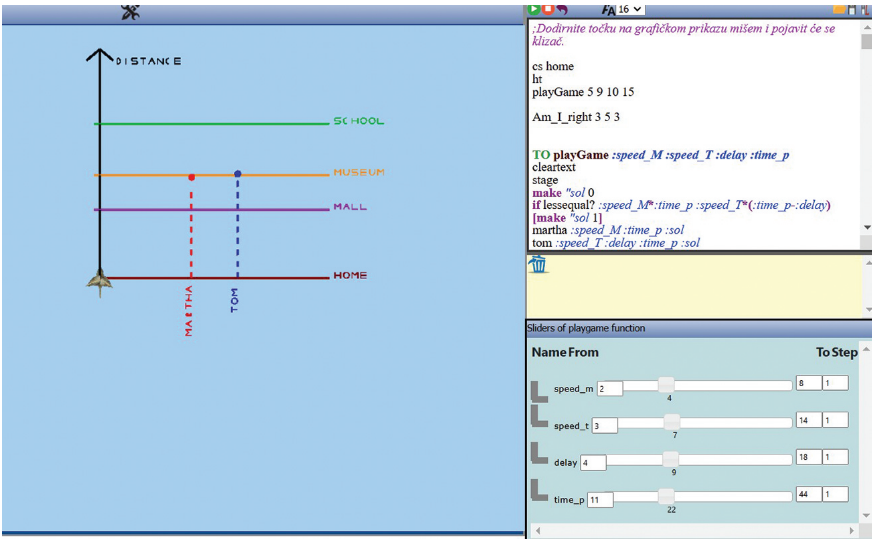

Figure 2. E(qua)scape game screenshot with graphical representation of movement in the left part of the screen, Logo language code in the upper right corner, and sliders for interaction with simulation in the bottom right corner.

The E(qua)scape game has several levels. It is necessary to solve the task correctly and with the help of the solution create a password that opens the next level. Students play the game by moving sliders in the simulation. That helps them to interpret and solve the assignment. Sliders represent parameters which are related to the time and speed and changing them affects the outcome of the assignment. Incorrect passwords will also appear when moving the slider. This prevents discovering passwords by accident.

6.2.3. Other GAMMA games

Other GAMMA games are listed in the table below (Table 2). More information on them can be found in the handbook and on official project page `http://www.project-gamma.eu/`.

Table 2. List of GAMMA games.

Name of the game	Mathematical topic	Short description
Do not blow up the balloon!	Derivatives	The Balloon game is a game for two players, playing against each other. The goal of the game is to control the size of the balloon by selecting appropriate rules for the magic pump – functions represented as formulas (in level Formulas) or graphs (in level Graphs), while avoiding that the balloon bursts or runs empty.
Hot Air Balloon	Direct and indirect proportion	The Hot Air Balloon game is an online game where players choose gas cylinders or valves to fill or release air from a balloon, competing with classmates or themselves. The goal of the game is to make as many choices as possible. The game can be played on computers, tablets, or mobile phones.
GeomWiz	Geometric shapes, angles of triangle, right angle, area of triangle, sine rule and cosine rule	GeomWiz is a single-player mobile game for learning and revising geometry, allowing players to move freely between levels with unique learning outcomes. Each level has rooms with questions for progress.
YoYo Bird	Trigonometric functions	Students play a game where they must choose a trigonometric function to control the bird's motion and the ball's motion. They must consider properties of trigonometric functions to gain points, avoid bird collisions, or make it difficult for the other player.
Function Dungeon	Linear function	Function Dungeon is a game where players navigate a labyrinth of rooms, often locked, to find hidden function-related problems. Interacting with objects and solving these problems opens up new rooms and unlocks the dungeon's secrets.

6.3. Snapshots from the Classrooms

6.3.1. ProbChallenge

Teachers in the project tested GAMMA ProbChallenge and its teaching scenario. Following the pilot, students and teachers provided feedback on the implementation. Students' comments revealed some deficiencies in the teacher's support. For example, some of students were confused about game mechanics because they did not obtain whole instructions on how to play the game. Reports also showed how important the teacher was to the lesson's success. The majority of the students who took part in the piloting described their learning as having "learned most of it, but need some clarification", which highlights the importance of teachers' support for DGBL. The instructor must encourage students to reflect on the gaming activity using whole-class discussion. Some teachers who piloted the game did not, however, reflect on game play activity.

6.3.2. E(qua)scape

Teachers, who participated in the project, piloted the game E(qua)scape and its teaching scenario. Following the pilot, teachers and students gave feedback on the implementation. Here we provide some issues. One teacher, according to reports, did not fully understand the goal of the E(qua)scape game: the goal is to apply previously learned concepts to new situations and the properties of systems of linear equations to real-world situations, not to learn how to solve a system of linear equations. Some students focused on the graphics of the MaLT2 environment rather than on meaning-making. The simulations give meaning to parameters in the system of linear equations and show how changing task conditions affect the system coefficients and, thus, the solution. Teachers who used the game said the material provided in the teaching scenario was sufficient for enacting the lesson. However, it is recommended that teachers prepare other tasks in which the system of linear equations is used so that students can apply their knowledge to other situations.

7. Conclusion

The goal of the education system is to incorporate innovation and change into the learning environment in order to equip students with the skills and resources necessary for the twenty-first century. Digital games are a tool for the development of twenty-first-century skills and a medium that motivates and encourages students to actively construct their knowledge (Hayak & Avidov-Ungar, 2023).

The work on GAMMA project shows how challenging it is to develop games that aid in the acquisition of high school mathematical knowledge. In the one hand, the game must be entertaining, while also requiring cognitive engagement from students. Cognitive engagement is the most reliable predictor of students' effective problem-solving actions aimed at their learning goal (Pan & Ke, 2023). Furthermore, incorporating modeling as learning support (scaffolding and worked-out examples) disrupts the game flow, i.e. the player's state of complete immersion in the game. There are also issues with games built on constructionist principles. To be specific, it may take the students a significant amount of time and effort to become acquainted with the environment. This issue, however, could be overcome with a Use-Modify-Create approach that scaffolds students (Kynigos & Grizioti, 2018).

Working on the project showed that the integration of games into mathematics lessons is a challenging endeavor. In addition to the numerous obstacles identified in the literature, the implementation of GAMMA games revealed that using games in the classroom is not intuitive for teachers. Professional development on DGBL is critical for effective DGBL implementation, but not in the usual form that contributes to teachers' professional growth, but in the form of teacher training. Our recommendation also stems from Callaghan et al.'s (2018) study, in which teachers expressed a desire for such DGBL support.

Digital games should not be used as stand-alone activities unrelated to overall mathematics instruction but rather as a component of a package of educational activities to achieve specific educational goals (Mayer, 2016). They should supplement not supplant teachers (Callaghan et al., 2018). This means that teachers must inform students on the game mechanics, interact with them during gaming, talk about the gaming experience, and discuss mathematics embedded in the game. If there are multiple levels of game to play, the gaming can and should be paused in order for the teacher to receive feedback on the aforementioned issues (Jukić Matić & Jukić Bokun, 2023). Moreover, teachers must inform students that this is a learning activity and it should not be used just for fun. We believe this element is very important, otherwise students will focus on the comparison of graphics and mechanics between entertainment games they play in their leisure time and the game given. And the goal of the DGBL lesson will not be achieved.

However, there are still unexplored topics on DGBL. One of them is developing and designing games to assist high school students in learning mathematical concepts. The second is related to the use of DGBL in mathematics classrooms. We believe that our project helped to resolve some of these issues.

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(Digitalno) učenje matematike temeljeno na igrama

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Sažetak. Za razliku od drugih STEM disciplina, matematika se često doživljava apstraktnom, odvojenom od stvarnog svijeta. Stoga učenici imaju problem kako da se užive i povežu s matematičkim sadržajima. Osim toga, tradicionalne metode poučavanja često ne uspijevaju aktivirati učenike i ne dopuštaju im samostalnost u kojoj su oni kreatori vlastitog razumijevanja. Učenje temeljeno na igrama (GBL), posebno učenje temeljeno na digitalnim igrama (DGBL), doima se kao obećavajući pristup učenju i poučavanju matematike upravo zbog svoje interaktivne prirode. Digitalne igre imaju jedinstvene karakteristike (pravila/ciljevi, senzorni podražaji, izazov, misterija, fantazija i kontrola) koje mogu povećati motivaciju za učenje i utjecati na promjenu učeničkog stava i ponašanja. U suvremenom obrazovanju, DGBL se pokazao kao učinkovit alat za učenje matematičkih koncepata i vježbanje matematičkih vještina. No, implementacija DGBL-a u nastavu predstavlja učiteljima popriličan izazov. U ovom pristupu, učitelji su dizajneri nastave, posrednici u igri ili organizatori rada. Stoga im treba pomoći u stjecanju znanja i vještina potrebnih za preuzimanje tih uloga.

Ključne riječi: učenje temeljeno na igri, digitalne igre, nastavnici matematike, GAMMA projekt

The Importance of Mathematical Knowledge in a Workshop with Advanced Programmers

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Abstract. On a sample of 12 fourth- and fifth-grade mathematicians and programmers from elementary schools in Osijek and its surroundings, selected by teachers as the best from their schools, a study was conducted on the correlation between the ability to solve mathematical and programming tasks. For this purpose, at the beginning and end of the education, the students took tests to determine their initial programming skills, mathematical thinking, and progress at the end of the school year. During the education, the educators monitored each student's motivation for solving problematic programming tasks about the knowledge of the mathematical background that was in that task. This research aims to determine if fourth- and fifth-grade students, who have good mathematical skills, can accurately and precisely model a problem while solving programming tasks.

The type of Educational research and conclusions are given in the paper.

Keywords: workshop, logical-mathematical thinking, programming, problem tasks, problem modeling

1. Introduction

At the beginning of the 21$^{\text{st}}$ century, with the advancement of technology, there was a need to promote STEM fields in the education system. With that goal in mind, in the 2018/2019 school year, Informatics was introduced as a mandatory subject in the 5th and 6th grades of elementary school, while in lower grades it was introduced as an elective subject.

To make it easier for teachers to explain the concept of programming and problem-solving tasks to students, there was a need to study the relationship between mathematical and programming knowledge, that is, there was a need to research whether previously acquired mathematical knowledge affects the mastery of programming skills. In recent years, several studies have been conducted that have yielded similar results, indicating that there is a positive correlation between students' mathematical knowledge and their success in learning programming languages.

In their research, Altin et al. (2021) studied how to integrate mathematics into programming lessons. They were interested in whether there was a difference in students' mastery of programming skills among those who learned programming through the integration of mathematics from those who had a classical approach to learning a programming language. The results of their research showed that students who learn programming through integration with mathematics acquire better programming skills, as opposed to students who learn programming in a traditional way. In their research, Liu et al. (2022) studied the influence of mathematics and student creativity on learning programming. Their results revealed that mathematical skills greatly influenced programming self-efficacy of programming through creativity. Therefore, Liu et al. (2022) suggest that, in addition to integrating mathematics into programming courses, teachers must also nurture and develop innovative thinking in students, as it is also crucial for easier learning programming and solving programming problems. In their research, Sofowora et al. (2022) examine whether students' mathematical skills affect their academic success in introductory programming. Based on the introductory programming test and the results of the students' national exit exams, they concluded that there is a positive correlation (and a very strong one) between the students' mathematical abilities and their success in mastering basic programming knowledge.

In the continuation of the paper, the research and education methods will be described to determine whether the results obtained the same (or similar) outcomes as in the previously mentioned research.

2. The introductory part of the research implementation

As part of the research group Advanced Education Methods, founded at the Faculty of Electrical Engineering, Computer Science and Information Technology Osijek (FERIT), a Group of Advanced Programmers was created, composed of students of the fourth and fifth grades of elementary schools in the city of Osijek and its surroundings. The workshop was designed and each meeting was led by two faculty teachers from FERIT specializing in the fields of mathematics and computer science.

In addition to the research mentioned above, the group's goal was also to improve the quality of programming knowledge in Osijek and to explore which of the advanced education methods give the best results in working with students. In order to be able to measure the progress in the students' knowledge, educators constantly checked the students' knowledge by solving tasks and successes in the school competition and the county-level competition.

At the beginning of the school year 2020/2021, educators informed principals and computer science teachers at elementary schools in Osijek and the surrounding area about the possibility of students from the fourth and fifth grades of elementary schools attending meetings of the Group of Advanced Programmers. The invitation was primarily addressed to the best mathematicians and programmers (the best according to grades) and to those students who are most interested in programming. The invitation stated that the meetings would take place once a week, for two school hours, and will be completely free for all participants. During the meetings, it was said; the participants will first be taught the basics of programming in the C++ programming language, and then the practice tasks from the computer science competition as preparation for the upcoming school competition. Although Python is the most commonly taught programming language in school, educators chose to teach the C++ programming language because educators wanted to teach young, advanced programmers how to create blocks placed within curly braces instead of text indentation (which is used in Python). Furthermore, the input and output commands of the C++ programming language (*cin* and *cout*) are easier to use than those same commands of C and Python.

The work with advanced programmers started in December 2020, and advanced programmers had classes four times in December 2020 and five times in January 2021, always in the evening, usually from 6:00 p.m. to 7:30 p.m., because then all the participants were free. Until the end of the school year, the advanced programming class was held once a week. The final meeting was held by solving various mathematical problems and puzzles (brain games) that can be implemented in solving programming problems.

In the group, 23 students enrolled and appeared at the first meeting, four from the fourth grade and 19 from the fifth grade of elementary school. After a few initial meetings, ten students completely gave up on further attendance, while all the others persisted in attending. The main reason for students' dropout was that they lacked motivation to further tackle challenges involving computer modeling of real-life problems due to their perceived (as they declared themselves) poor prior knowledge and fear of failure.

3. Methods

The goal of the first meeting was to assess the initial knowledge of the participants in the advanced programmers' meetings. For this purpose, the project leader designed an initial ("initial") test that included the following tasks:

1. Write a program that will print the smaller of the entered numbers *a* and *b*.
2. Write a program that will print the perimeter and area of the square for the given length of one of its sides.
3. (A task from a school competition in the category of algorithms for 5th-grade elementary schools, Basic/C/C++/Pascal from 2012, `https://informatika.azoo.hr/natjecanje/dogadjaj/151/rezultati`

Gjuro XIII, the king of lemurs from Madagascar, enacted a special law regarding the determination of the total value of purchased goods in the shops of his kingdom. Namely, in his kingdom, there is no banknote/coin of **one lem** *(***lem** *is like a* **kuna** *in our world). Therefore, every issued bill for purchased goods whose value is* **not divisible by five** *could not be charged. That is why Gjuro XIII decided that the value of such bills should be rounded to the nearest whole number divisible by five. Help the king write a program that will enforce his law in practice.*

(*Mathematical/programming explanation of the task:* In this task, the goal is to write a program that will print the closest multiple of 5 for the entered natural number.)

In this first meeting, the participants had half an hour and were free to use any programming language (even though the goal of this group was to work exclusively in the programming language C++, educators believed that students might have greater knowledge of the programming languages such as Python, Logo or C).

In the mentioned initial exam, none of the 23 participants solved any of the given tasks. The participants knew how to access the *OnlineGDB* website or how to access *DevC++* or similar programs on their personal laptops, but they mostly did not know how to start programming. Although participants did not know how to write a program to solve the given problems, they knew mathematical formulas and could logically explain what they needed to do.

Since the participants did not know the basics of the programming language or the syntax, educators, therefore, decided to start the meetings with a programming school in C++, from the very beginning, from the basics, in order to learn, first, the syntax of writing programs.

After just two weeks, all students mastered the use of the C++ programming language environment and the basics of input and output commands. They became familiar with variables and their types and could independently make their own calculator that performs the four basic operations on variables of integer and float type. After three meetings, they also mastered the use of the "if" command.

Most of the tasks in school competitions for fifth-grade elementary school students can be successfully solved even without knowing the loops, so the students who participated in Advanced Programmers meetings had been solving simpler competition tasks on their own since mid-December.

In January 2021, the students mostly prepared for the school competition, without learning any new tools in programming (except for introducing the *mod* operator, i.e. the % sign, for solving purposes) and used different types of variables.

At the school competition held in early February 2021, on the ranking list of Osijek-Baranja County, the students attending the course won the 1st, 3rd, 4th, 5th, 8th, and 10th place (Competition in informatics, Algorithms, School level, n.d.). That means that out of six students who applied for the competition, all six were among the top ten, which educators considered a success, especially considering the initial knowledge at the beginning of December of that school year.

All six students advanced to the further stages of the competition and participated in the county competition, which took place in March 2021, and they won the 1st, 2nd, 7th, 8th, 9th, and 10th place (Competition in informatics, Algorithms, County level, n.d.). However, the students who won the 7th, 8th, 9th, and 10th place did not score any points.

None of the students from Osijek-Baranja County advanced to the national competition due to a low number of points.

4. Motivation

The primary tool for monitoring motivation was (in addition to regular attendance tracking) continuous visits to students at their workplaces and monitoring their approach to the assigned task (the task was most often taken from previous informatics competitions). Furthermore, after the educator confirmed that the task was done well, the student put a plus sign next to his or her name, so these plus signs on the board were used as an excellent indicator of motivation.

Regarding the motivation for attendance and active participation in classes, all students (except for the ten who dropped out at the beginning) regularly attended the Advanced Programmers meeting in December 2020 and followed the programming classes and competition task-solving with full concentration. They quickly progressed and showed a strong interest in the subject matter.

In December, during the first few meetings, the students were highly motivated. The atmosphere was productive throughout the entire ninety minutes, and the dynamics of the class were at a high level.

In order to increase student motivation at the beginning of December, educators used the seriousness of the situation, the possibility of mastering new topics, and the importance of programming and algorithms competitions as motivational tools. The lessons had a fixed structure, with mandatory roll call and attendance sheets. The discipline was higher because the other students were unknown to the participants, and they respected them more, showing greater obedience towards unfamiliar teachers. Additionally, discipline was greatly influenced by the fact that the students easily mastered the basics of programming and thus easily solved the problems that were put in front of them. This developed a desire among the students to be the first to solve a task, leading them to be fully concentrated on fulfilling their duties.

In January 2021, student motivation decreased, which was evident from the fact that some students sometimes secretly played computer games, some talked to each other more often about topics not related to classes, and some texted on their cell phones. During the period from January 7th to 21st, the students solved the least amount of tasks independently, although those were the most important meetings as educators told the students how to solve tasks in preparation for the school programming competition. More precisely, in several initial meetings after the winter holidays, almost none of the students solved any of the tasks independently. However, as the end of January approached and the school competition that was held at the beginning of February approached, the students began to pay more attention to the lessons and play less on computers and mobile phones.

Table 1. Attendance to classes by advanced programmers by weeks.

	The average number of attendees	**Concentration and motivation**	**Progress**	**Explanation**
December 2020	18	Excellent (all students participate in the work and attempt to solve all assigned tasks)	Excellent	New environment, new teachers, the seriousness of the situation, competition, desire to stand out, responsibility towards the parents who bring them, and fear of failure.
January 2021	14	Unmotivated (students do not attempt to start solving the task without the involvement of the educator)	Very low	The motivation somewhat decreased, but the concentration decreased to a greater extent, especially in the first few meetings after the winter holidays.
February 2021	12	Good (students are active in solving)	Very low or non-existent	Motivation exists, students who understand the beauty of programming and want to improve in it come regularly, but their concentration is low. There is very little progress in solving competition tasks.
March 2021	11	Good (students are active in solving)	Very low	The motivation is present, but the concentration is not. Additionally, there is very little progress in solving competition tasks.
April 2021	10	Good (students are active in solving)	Very low	Motivation exists. The concentration is so low that the meetings have to be divided into two parts. In the first hour of class, students are in a standard classroom with instruction on the school board while students come up with algorithms and write programs in notebooks.

The tools educators used to try to increase student motivation for work from mid-January to early April were an almost friendly relationship between teachers and students, the freedom to communicate with any other participant in class, more independent work at the beginning of solving the task (during the algorithm design period, and before starting to write program code), and globally less strictness and discipline in classes.

Table 1 shows the progress of the participants as well as their motivation for learning and solving programming tasks, considering the number of solved tasks and student behavior and approach to solving.

On the days when the students were not concentrated enough, an interesting phenomenon occurred where they were unable to solve the tasks that they had previously solved in one of the previous meetings. The lack of concentration was so profound that it seemed like not only had they not progressed or stayed at the same level as in previous meetings, but it was evident that their knowledge had regressed due to the decline in concentration.

In April, there was a need to change the usual way of working because the previous way of working in front of computers was not effective, as explained above. For this purpose, educators changed the course of the class and held the first 45 minutes in a traditional way, using a blackboard and chalk, with students writing in notebooks, without computers. Educators first solved the given three problems without a computer and students wrote the basic algorithmic ideas in a notebook. After that, students and educators would move to the computer classroom and solve all the given tasks on the computer, i.e. only then would the students write the programming code independently.

It turned out that such a class was somewhat better than usual for better students, meaning that better programmers used those initial 45 minutes to think about how to create an algorithm, and then in the second 45 minutes they programmed it on the computer. This showed that the problem of how to better focus students on thinking was partially solved.

Less successful students, although guided by the teacher in the first 45 minutes in the traditional classroom according to the ideas they would use to create the program, were not able to independently complete the program either in their notebooks or on the computer later.

Overall: Teaching in two stages (the first part focused on thinking and creating algorithms in the mind or using paper and pencil, and the second part on the computer) helped better programmers and mathematicians achieve good results. Weaker students were as successful as usual or slightly more successful.

This leads to the conclusion that students who have a sense of modeling problems for computer competitions can go further and expand their limits with better teaching methods, while students who are weaker in programming thinking will not be able to go much further in their abilities even with different forms of teaching. In other words, weaker programmers can learn tools, but they cannot reach a level of knowledge where they would achieve greater success, for example, at county or state competitions, with special types of teaching.

One of the ways in which the teachers tried to increase student motivation for work was by first writing the name of the student who correctly solved the task on the board. Namely, when each student finished the task and received confirmation from the teacher that they had solved the task correctly, they went to the blackboard and wrote a plus sign next to their name. This method, especially for some students, produced good motivating results.

5. Mathematics and programming

Five students from this programming group went to the math competition, two of them from the fourth grade and three from the fifth grade of elementary school. These same five students solved the largest number of tasks in the Group of Advanced Programmers. It is important to note that fourth graders often solved the tasks from the fifth-grade competition before and better than other fifth-grade students who were one year older than they were.

Students who are able to model problems from the external world using mathematical tools (i.e., they are able to set up equations or determine the given and desired quantities based on the text of the problem and use formulas, etc.) are also able to model such problems using computer programming tools. That is, those students know what the input and output variables are and how to structure a program to obtain the desired result.

The best student in the group was also the best mathematician and programmer in the group, i.e. he achieved the best results in both mathematical and programming competitions.

At the final meeting of the advanced programmers' group, educators gave the students a written math test consisting of five tasks, with each task carrying 20 points (two tasks in which the known and required sizes of geometric figures were given and three text tasks with the number of years, pouring liquid and workers who complete part of the work in a certain number of days). The results were as expected - the students who knew how to model sentences from the outside world with a program in the C++ programming language were able to model a problem task with mathematical equations and formulas.

Table 2 shows the achievements that the participants have made in competitions in mathematics and programming competitions, as well as on the test that were given to them in the Group of Advanced Programmers. As mentioned earlier, fourth-grade students achieved excellent results on both the mathematics and programming test, but since there were no programming competitions in the fourth grade of elementary school, they could not participate in them. The table contains only the results of students who attended regularly throughout the school year.

Table 2. Student success in our test, mathematics and programming competitions.

Student	**Attending**	**Success in the initial test at the beginning of the school of advanced programmers**	**Success in the math test**	**Mathematics competitions**	**Success in the developer test at the end of the school of advanced developers**	**Programming competition**
Student 1 (fourth grade)	87 %	0 %	90 %	School competition County competition	100 %	–
Student 2 (fifth grade)	82 %	0 %	85 %	–	100 %	School competition: 5th place in the county
Student 3 (fifth grade)	92 %	0 %	85 %	School competition	90 %	School competition: 4th place in the county
Student 4 (fourth grade)	95 %	0 %	85 %	–	80 %	–
Student 5 (fifth grade)	82 %	0 %	80 %	–	Didn't take part	School competition: 3rd place in the county County competition: 2nd place
Student 6 (fifth grade)	82 %	0 %	15 %	–	70 %	School competition: 10th place in the county
Student 7 (fifth grade)	72 %	0 %	15 %	–	70 %	–
Student 8 (fifth grade)	82 %	0 %	13 %	–	75 %	–
Student 9 (fifth grade)	77 %	0 %	Didn't write	School competition	Didn't take part	School competition: 8th place in the county
Student 10 (fourth grade)	77 %	0 %	Didn't write	School competition	Didn't take part	–
Student 11 (fifth grade)	72 %	0 %	Didn't write	–	Didn't take part	–
Student 12 (fifth grade)	95 %	0 %	Didn't write	School competition County competition	Didn't take part	School competition: 1st place in the county County competition: 1st place

6. Adopting abstract concepts in programming

One of the most difficult problems in programming was the transition from common quantities such as numbers or text, to variables that *represent* numbers or text.

It is interesting that the students learned to use the *"if"* command or the *"for"* loop faster than using variables to solve a specific task.

The conditional statement (only *"if"* statements were covered) was learned in two to four school hours and students knew how to use them.

The students mastered the loop (only the *"for"* loop was processed) in four to six school hours. However, the variables were the most difficult for them and they were not able to fully grasp them even after ten hours of learning and practice. The difficulty in mastering the concept of variables is logical because the students are of that age, i.e. from the 4th or 5th grade of elementary school, when they had not yet fully understood the abstraction of replacing mathematical quantities with letters.

Therefore, in order to better understand the use of variables, the so-called *step-by-step tasks* were used, such as:

1. Print all the numbers between the numbers 11 and 137.
2. Print all the numbers that are between the numbers 11 and 137 that give a remainder of 2 when divided by 5.
3. Write a program that will print all numbers between natural numbers a and b that give a remainder of 2 when divided by 5.
4. Write a program that, for the entered natural numbers a, b, c, and d, will print all the numbers that are between the numbers a and b and that, when divided by c, give the remainder d.
5. Write a program that, for the entered natural numbers a, b, c, and d, will print all the numbers that are between the numbers a and b that, when divided by c, give the remainder d, and will then print
   a) how many such numbers there are, and,
   b) what is the sum of such numbers.

All participants were able to solve points 1, 2, and 3 of the above-listed step-by-step tasks, but only the best mathematicians were able to solve tasks 4 and 5 with some help, as they were the only ones who could understand the concept of variables at that moment. Interestingly, when the same (or similar) variable tasks were given to the students as a review at the next meeting, they were unable to solve them independently, which indicates that the concept of introducing variables in the "for" loop had not been fully mastered.

Within the research group Advanced Education Methods at FERIT, the best student programmers from the faculty and high school students from the city of Osijek have been meeting every week for several years. During these meetings, they solve interesting and current programming problems and prepare for programming competitions (such as IEEExtreme, STEM Games, etc.). Through working with the best programmers, it has been observed that in further programming education, mathematics is not a necessary prerequisite for mastering programming problems. Moreover, the best programmers are not necessarily the best mathematicians in higher education. In other words, it has often been shown that the best programmers have problems with passing math courses.

However, some students achieved excellent grades in mathematics courses as well as excellent results in programming competitions. From this, it is visible that the mathematical knowledge acquired at the university does not necessarily help students learn programming subjects, but some students are successful in both areas. The conclusion is confirmed by a study by Razak and Ismail (2018).

We are aware that measuring motivation precisely is challenging, so in future research, we will attempt to introduce more accurate and precise methods of measuring motivation. Furthermore, we are aware that due to a relatively small number of participants at our school, the results may not be representative. For future research, it would be much more effective to have two groups of participants, with whom we would work in different ways, and then compare the results (which was our intention that could not have been realized due to an insufficient number of registered participants).

7. Conclusion

During one school year, as long as the gathering of the Group of Advanced Programmers lasted, the quality of programming knowledge among fourth and fifth-grade elementary school students was raised, which was also shown by the results at school and county competitions where the participants achieved the best places. Additionally, research has shown, as in Altin et al. (2021), Liu et al. (2022), Sofowora et al. (2022) that students with well-developed logical-mathematical thinking are more successful in modeling and solving programming problems presented to them. The reason lies in the fact that the tasks in computer science competitions are closely related to mathematics, meaning that the problems are reduced to modeling real-life mathematical problems. Therefore, in order for fourth- and fifth-grade elementary school students to successfully solve a competition task, they must first approach the problem mathematically, i.e., first solve the problem on paper, and then implement it in the programming code. In other words, without a mathematical approach and analysis of the problem, students of the 4^{th} and 5^{th} grades of elementary school will find it difficult to solve the programming tasks that are put before them.

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Važnost matematičkog znanja u radionici s naprednim programerima

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Sažetak. Na uzorku od 12 matematičara i programera petih razreda osnovnih škola grada Osijeka i okolice, odabranih od strane nastavnika kao ponajboljih iz svojih škola, napravljeno je istraživanje o korelaciji mogućnosti rješavanja matematičkih i programerskih zadataka. U tu svrhu, na početku i na kraju edukacije učenici su rješavali ispit kako bi se utvrdilo njihovo početno znanje iz programiranja, ali i matematičkog razmišljanja te njihovo napredovanje na kraju školske godine. Tijekom edukacije, edukatori su pratili kod svakog polaznika njegovu motivaciju za rješavanjem problemskih programerskih zadataka s obzirom na poznavanje matematičke podloge koja se nalazila u tom zadatku. Cilj istraživanja je utvrditi mogu li učenici četvrtih i petih razreda, koji su ujedno i dobri matematičari, točno i precizno modelirati problem prilikom rješavanja programskih zadataka.

Oblik edukacije istraživanja i zaključci su dani u članku.

Ključne riječi: radionica, logičko-matematičko razmišljanje, programiranje, problemski zadaci, modeliranje problema

Digital Pedagogy and Teaching Mathematics – Trends, Perspectives, Limitations and Challenges

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Abstract. In recent years, the increased use of modern digital technologies in the teaching process has, among other things, accelerated the development of new areas of pedagogy. In education research, the field of *digital pedagogy* is increasingly represented, and it is shown that there is a significant potential for its application in the teaching of mathematics, which is also supported by numerous examples of recent practice of teaching and learning mathematics. The paper clarifies the concept of digital pedagogy and its potential in solving the traditional challenges of teaching mathematics, as well as the possibility of developing new strategies and approaches to the design and development of an innovative learning process.

Various possibilities of using digital technologies that teachers can use in efforts to improve the quality of the teaching process are suggested, and the shortcomings and some new challenges brought by the increased use of ICT in mathematics teaching are highlighted. The application of digital educational technologies in the teaching of mathematics and the changed role of the mathematics teacher are critically reviewed, as well as the knowledge and skills the teacher needs in order for students to learn mathematics more efficiently and develop optimally through the various forms of formal, non-formal, and informal learning.

Keywords: digital pedagogy, teaching mathematics, teacher competencies

1. Introduction

During the last two decades, new technologies have distinctly influenced the way children and adolescents communicate with others, how they learn and how they live their lives. More recently, due to the SARS-CoV-2 virus pandemic, the use of information and communication technology in education has recorded an exponential growth despite the slowly changing educational systems. Therefore, the challenges of traditional teaching and learning methods and the possibilities of introducing more innovative forms of teaching have become even more relevant. As a result, the implementation of digital technologies in the educational system has become part of educational policies that aim to train highly qualified individuals who are able to adapt to the conditions of modern life and the job market.

The use of digital technologies in education has so far mainly been limited to higher education and various online courses, as well as for high school students who, due to geographical or other reasons (special educational needs, elite athletes, etc.), could not regularly attend classes. The increasing prevalence of digital technologies in education has led to the need to redefine traditional teaching and learning methods and introduce innovative forms of teaching that would be more suitable for the new generation of students described as digital natives (Jones et al., 2010; Clark et al., 2015).

In recent years, there has been increasing talk about digital pedagogy as a new branch of pedagogical science that focuses on the changes in the educational process caused by the increasingly frequent use of digital technologies, and supports innovative ways in which students and teachers can organize their activities and create new didactic models and approaches to teaching. The concept of digital pedagogy has gradually evolved as the field of research itself has expanded to meet the needs and demands of educational systems. Croxal (2012) within digital pedagogy refers to the use of electronic elements to enhance or change the experience of education, while Kivunja (2013) presented a field of "digital pedagogy" as the skill of embedding digital technologies into teaching so that they enhance learning, teaching, assessment, and curriculum. Therefore, Garber-Pearson and Chin Roemer (2017) state that digital pedagogy must include a critical component through which a teacher can evaluate how and why to use a particular technology to improve the teaching process. For Väätäjä and Ruokamo (2021), digital pedagogy refers to the pedagogical use of digital technologies so they presented a model of digital pedagogy which comprises of pedagogical orientation, pedagogical practices, and digital pedagogical competencies that should explain key dimensions of pedagogical use of digital technologies. The field of digital pedagogy raises key questions about the changing role of teachers and students, student motivation for learning in an interactive environment, the development of digital competencies, as well as issues related to growing digital inequality and psychological impacts of technology on students' daily lives.

Regardless of contemporary methods of teaching and the use of ICT in mathematics education, students must acquire basic mathematical concepts and computational and algebraic procedures in order to master mathematical content. Adequate use of ICT in mathematics education has exceptional potential for learning and teaching mathematics, as mathematical content mediated through media and mathematical applications and programs has increasing opportunities to make mathematics more accessible, interesting, and understandable to students. The main goal of this paper is to present the possibilities that existing digital technologies offer for mathematics education and to consider how to improve the quality of mathematics education by using ICT, and whether (and to what extent) it can make learning mathematics easier and more accessible to all students in order to enhance their learning experiences.

2. Digital Pedagogy and mathematics education

There are various recent challenges in mathematics education such as: curriculum and standards, equity and access, teacher professional development, online learning, use of ICT etc. In addition to those challenges new generations of students, are having short attention span, lack of motivation, diverse learning needs and are too dependent on calculators and other tools. With the shift towards online learning due to the COVID-19 pandemic, teachers and students have had to adapt to new ways of instruction and learning. This has presented challenges such as difficulties in maintaining engagement and interaction, the need for new teaching strategies and tools, and the various factors that can impact access to technology and resources.

Under the context of teaching, Wempen (2014) highlights information and communication technologies (ICT) as tools such as: email, social networks, various forums and blogs, communication through mobile devices, and a range of different forms of internet communication (according to Matotek, 2015). On the other side, for the teaching mathematics, it is more relevant to focus on software packages and interfaces designed for educational purposes to master the teaching content. In Croatia, the Moodle learning management system was the first digital platform to stand out in our schools as a tool for creating electronic educational content and conducting remote teaching, where software packages as Mathematica, Geogebra, and Geometers Sketchpad helped practitioners to widen their digital resources in specific curriculum domains which combining with Moodle and other recent digital platforms and applications can answer a wide range of challenges and critical issues in mathematics education. Moodle is an acronym for modular object-oriented dynamic learning environment. Moodle has over 100,000 users at all levels of education in Croatia, and is available for use by all students and teachers through CARNet (Croatian Academic and Research Network). Moodle has several ways it can be used for teaching and learning, such as a database of learning content, additional materials for those who want to learn more, knowledge tests and quizzes, and content for self-study (Matotek, 2015). The advantage of Moodle is that learning materials and activities are available to students at home or anywhere they have access to a computer and the internet (Pavleković et al., 2010). In 1999, the well-known methodology expert M. Pavleković spoke in her book "Methodology of teaching mathematics with informatics" about the confrontation of traditional and modern methods in teaching mathematics. In the same book, she spoke about the multiple applications of the *Mathematica* software package as a teaching tool and method, which was explicitly demonstrated through numerous applications in solving problems and more complex geometric theorems. On the other hand, the software packages Geogebra and Geometers Sketchpad are primarily designed to work in the classroom with the aim of providing a clearer and more visual representation of certain teaching content that is difficult to show on the board. They allow the creation of dynamic constructions and all mathematical objects that can be moved, their properties changed, and mathematical relationships investigated.

These programs enable the implementation of the principle of visuality in teaching mathematics, which is the basis for understanding geometry. As for geometry classes, Geogebra and Geometers Sketchpad have completely replaced Mathematica due to their simplicity of use, especially since students can easily use them independently. Constructions can be performed in minimal time, while the drawings are clear and can be dynamically changed according to all mathematical principles used in geometry.

The trend of video lectures in Croatia began some ten years ago when perhaps the most popular mathematics teacher, Toni Milun, started recording the process of solving some problems from his classes at the request of his students. This resulted in an increasing number of views of video lectures or video clips on popular internet networks, in which the procedures for solving mathematical problems are explained or repeated. Over time, a knowledge portal was created, which provides video clips in which mathematics teachers explain and explicitly solve problems from almost all mathematical chapters, starting from elementary and high school. Today, the same portal has more than ten million views, but what is even more important is that the whole idea, due to its popularity among students, has spread so that other teachers also post solutions to problems from their classes on the internet.

Over the last two decades, there has been a growing number of applications that are being used to enhance learning mathematics for students. Some of the most popular ones include:

Photomath: A tool that allows students to take a picture of a math problem and get an instant solution, along with step-by-step explanations. Photomath can be applied to: arithmetic (basic operations as addition, subtraction, multiplication, division), algebra (systems of equations, solving linear equations and inequalities, polynomials and factoring), geometry and trigonometry (basic geometric figures and properties, trigonometric ratios and functions), calculus (limits and continuity, derivatives and integrals), discrete mathematics, statistics and probability (sequences and series, mathematical induction, descriptive statistics, probability calculations and combinatorics), etc.

Quizlet: A free online tool that allows students to create flashcards and study sets for math concepts, formulas, and vocabulary. URL: `https://quizlet.com/`

Quizlet, an online learning tool that allows users to create and study custom flashcards and other learning activities, can be applied to various mathematics curriculum domains. The main curriculum domains to which Quizlet can be referred are similar to one in Photomath and can be equally applied in arithmetic, algebra, calculus, geometry and trigonometry, discrete mathematics, combinatorics and statistics, number theory etc.

Mathway: A tool that provides step-by-step solutions to mathematical problems, including algebra, calculus, and statistics. It also offers a graphing calculator and practice problems. URL: `https://www.mathway.com/Algebra`

Wolfram Alpha: A computational knowledge engine that can answer a wide range of mathematical questions, from basic arithmetic to complex calculus problems. Wolfram Alpha is a powerful tool for exploring wide range of mathematics curriculum domains, providing step-by-step solutions, visualizations, and detailed explanations for various mathematical problems.

Prodigy: A game-based learning platform that offers math practice and assessments for students in grades 1–8. URL: `https://www.prodigygame.com/main-en/`

Prodigy covers a broad range of mathematics domains primarily aimed at elementary and middle school students. It is game-based approach which is particularly effective for engaging students in practicing and mastering foundational mathematics skills through interactive and adaptive gameplay. It enhances learning process by explaining mathematical thinking clearly and coherently, solving problems related to time, money, and real-world scenarios, solving problems involving ratios, rates, and proportions, plotting points and understanding the coordinate plane, identifying and classifying shapes and their properties, understanding and using place value in different contexts to help students develop and improve number sense etc.

Matific: A gamified learning platform that offers interactive math activities and games for students in grades K-6. URL: `https://www.matific.com/hr/hr/home/`

Matific, an educational platform offering interactive math activities and games, can be applied to activities focused on addition, subtraction, multiplication, and division, games and exercises for understanding, comparing, and performing operations with fractions, decimals, and percentages, Games that involve recognizing and extending number patterns and sequences, exercises to develop skills in estimation and rounding numbers, reading and interpreting different types of graphs and charts and many other activities that can help students build a strong foundation in mathematics through hands-on learning and problem-solving experiences.

These applications can be used by students to supplement classroom instruction, practice and reinforce math concepts, and gain a deeper understanding of mathematical principles. They can also help students work at their own pace and provide personalized feedback and immediate support and can allow students to access educational content from anywhere at any time.

2.1. Advantages and disadvantages of using digital technologies in mathematics education

Although digitization has brought many benefits, such as reducing disparities in access to education, developing standardized forms of measuring educational achievements and international comparisons, and actively involving students in the learning process, it has also generated certain drawbacks. Among them, aspects such as reduced teaching quality, the creation of a digital divide and inequalities among students, and an increase in psychosocial problems for teachers and students have become particularly prominent during the pandemic. Research conducted by Greenhow et al. (2021) has shown that the wrong choice of digital tools in pandemic teaching has resulted in lower engagement among disadvantaged students, especially those who lack access to devices and the internet. In most cases the implementation of digital technologies does not result in a change in established teaching practices and recent international research supports this claim, showing that, although teachers have a positive attitude towards new technologies (Serin & Bozdağ, 2020), the frequency of their usage remains low (OECD, 2019).

But using technology in the classroom in proper way can enhance learning experiences (Higgins et al., 2012). Technology can provide access to a wide range of educational resources such as videos, simulations, and interactive activities (Hew & Brush, 2007b; Eom & Ashill, 2016). Technology can also be used to create personalized learning experiences that meet the needs of individual students (Pane et al., 2015). Adaptive learning software can assess students' abilities and adjust instruction to help them master concepts at their own pace (Knewton, 2012; Pane et al., 2015). Technology can be used to facilitate collaboration among students. Online discussion forums, video conferencing, and other collaborative tools can help students work together on group projects or share ideas and feedback (Hrastinski, 2008).

Technology can make learning more accessible for students with disabilities with assistive technologies such as screen readers, captioning, and audio descriptions can help to make content more accessible to students with visual or hearing impairments (Alnahdi, 2014). Online quizzes and assessments can provide immediate feedback to students, allowing them to identify areas where they need more practice or review and help them to track their progress and improve their understanding (Nicol & Macfarlane-Dick, 2006).

The increased use of ICT (Information and Communication Technology) in mathematics teaching has brought about several challenges, including:

— Technical issues: The use of technology in the classroom can lead to technical issues, such as software malfunctions or connectivity problems, which can interrupt the learning process and cause frustration (Hew & Brush, 2007a; Al-Zaidiyeen et al., 2010).
— Lack of teacher training: Teachers may not be adequately trained in the use of ICT tools and resources, which can hinder their ability to effectively integrate technology into their teaching practice (Bingimlas, 2009).
— Over-reliance on technology: Over-reliance on technology can hinder students' ability to develop critical thinking and problem-solving skills, as they may become too reliant on technology to solve problems (Säljö, 2010).
— Inequitable access: Not all students may have access to the same technology or resources, which can create an inequitable learning environment and limit some students' opportunities for learning (Wresch, 1996; Warschauer, 2004).
— Distractions: Technology can be a source of distraction for students, as they may be tempted to browse social media or other non-academic websites during class time (Kraushaar & Novak, 2010; McCoy, 2013).
— Student engagement: While technology can be a valuable tool for enhancing student engagement, it can also lead to disengagement if not used effectively. Teachers must carefully select and use technology tools that are engaging and aligned with students' learning needs and preferences.

Overall, the increased use of ICT in mathematics teaching has brought about several challenges that must be carefully managed to ensure that technology is used effectively and in a way that enhances students' learning experiences. Digitalization is accompanied by a change in the traditional paradigm of upbringing and education, where the emphasis shifts from the teacher to the act of learning, that is, the student, and the focus moves from acquiring knowledge to constructively creating knowledge (Frolova et al., 2020).

3. Future perspectives of digital pedagogy in mathematics education

Digital pedagogy refers to the use of digital technologies in the design and delivery of educational content and instruction. It involves the integration of digital tools and resources, such as online platforms, multimedia materials, and educational software, into the teaching and learning process. Amin (2016) emphasizes that it is evident that we do not necessarily need applications, gaming platforms, or digital aids for more effective learning. Although they can be helpful, we must primarily reconsider the role of the teacher and the knowledge they possess in relation to the content and resources available to students on the Internet. This re-evaluation aims to assist students more effectively in learning how to learn, how to think and reflect on acquired knowledge, and how to apply it efficiently, thereby contributing to their own development through various formal, non-formal, and informal forms of learning, whether at school or at home on a computer or when exploring information of interest together with friends

Here are some strategies for efficiently using technology in teaching mathematics:

— Choose appropriate technology that best suits the mathematical concept being taught. Some technology tools that can be used include interactive whiteboards, graphing calculators, spreadsheets, and math software.
— Provide clear instructions to students on how to use the technology tool being used. This will help students to focus on the mathematics rather than struggling with the technology.
— Integrate technology into math lessons in a meaningful way. Use technology to enhance, not replace, traditional teaching methods. For example, use graphing calculators to explore graphing functions and understand transformations.
— Technology can help to engage students in math by making it more interactive and dynamic. Use technology to create virtual manipulatives, simulations, and interactive games that help students to visualize and understand mathematical concepts.
— Encourage collaboration by using technology tools such as interactive whiteboards or online discussion forums. This will allow students to work together and share their ideas and approaches to solving problems.
— Technology can be used to provide students with opportunities for practice and feedback. Use online quizzes, adaptive learning programs, and math software to provide students with immediate feedback on their progress.

— Use technology to monitor students' progress and adjust instruction accordingly. Use online tools to track student progress and provide personalized feedback and support.

By following these strategies, technology can be used efficiently and effectively to teach mathematics and enhance student learning. This primarily involves an environment where students are encouraged to ask questions, gather information, and create new knowledge that is relevant to their personal lives in the context of living in the digital era of the 21st century (Avidov-Ungar & Forkosh-Baruch, 2018).

The role of the modern mathematics teacher has changed significantly due to the increased use of technology in education and changes in educational pedagogy. Mathematics teacher is no longer the sole source of knowledge and information. Instead, the teacher's role has shifted towards being a facilitator of learning, guiding and supporting students in their exploration and understanding of mathematical concepts. With the increasing use of technology in education, the modern mathematics teacher must be able to effectively integrate technology into their teaching practice. This involves being familiar with various types of educational technology and how to use them to enhance the learning experience of students. Mathematics teacher must be able to differentiate instruction to meet the diverse learning needs of students. This involves using a variety of instructional strategies, materials, and assessments to cater to different learning styles and abilities and using assessment data to identify student needs, adjust instruction to address those needs, and evaluate the effectiveness of instructional strategies.

4. Final considerations

Within the possibilities of modern technologies, social trends, and the specific characteristics of 21st-century students, all of recent educational challenges must be considered within the framework of digital pedagogy and its potential to improve the quality of mathematics teaching and establish new paradigms of mathematics education. Lewin and Lundie (2016) emphasize that digital pedagogy is not just about a static list of tools but also encompasses fundamental values and interests, goals and strategies, combining the philosophy of technology and information theory with critical pedagogy and educational philosophy.

To improve the quality of mathematics teaching, it is necessary to differentiate the approach and use of digital technologies based on the students' different school ages. For example, with primary school students, it would be more appropriate to use various games and quizzes to engage and motivate them in learning. In middle and high school, the focus should shift towards using different computer programming interfaces to present and learn content, as well as to facilitate information delivery in a way that is more effective in achieving the students' educational outcomes according to their educational needs. This may include using online discussions, interactive simulations, virtual reality environments, and other digital tools to create more dynamic and immersive learning experiences.

Digital pedagogy emphasizes the importance of leveraging technology to enhance the learning experience and engage students in new and innovative ways. Effective digital pedagogy also involves careful planning and consideration of how technology is used to support learning outcomes and foster critical thinking, creativity, and collaboration among students. It involves a shift from traditional teacher-centered approaches to more student-centered, active learning strategies that promote inquiry, experimentation, and discovery. Therefore, the role of the modern mathematics teacher has shifted towards being a facilitator of learning, an integrator of technology, a differentiator of instruction, a collaborator and communicator, and a data-driven decision maker. These changes reflect the evolving nature of education and the need for teachers to adapt to meet the changing needs of their students and help them to enhance learning, increase their engagement, and provide them greater access to educational resources.

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Digitalna pedagogija i nastava matematike – trendovi, perspektive, ograničenja i izazovi

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Sažetak. Povećana upotreba suvremenih digitalnih tehnologija u nastavnom procesu posljednjih godina inicirala je, među ostalim i ubrzani razvoj novih područja pedagogije. U istraživanjima odgoja i obrazovanja sve je više zastupljeno područje *digitalne pedagogije* te se pokazuje kako postoji značajan potencijal primjene u nastavi matematike, kao i brojni primjeri recentne prakse učenja i poučavanja matematike. U radu se pojašnjava koncept digitalne pedagogije i njezin potencijal u rješavanju tradicionalnih izazova nastave matematike te mogućnosti razvijanja novih strategija i pristupa oblikovanju i razvoju inovativnog procesa učenja.

Sugeriraju se različite mogućnosti korištenja digitalnih tehnologija koje nastavnici mogu koristiti u nastojanjima unapređenja kvalitete nastavnog procesa te se ističu nedostaci i neki novi izazovi koje sa sobom donosi povećana uporaba IKT-a u nastavi matematike. Kritički se preispituje primjena digitalnih obrazovnih tehnologija u nastavi matematike te promijenjena uloga nastavnika matematike, kao i znanja i vještine koje su mu potrebne kako bi učenici efikasnije učili matematiku te se optimalno razvijali kroz različite formalne, neformalne i informalne oblike učenja koji su im danas na raspolaganju.

Ključne riječi: digitalna pedagogija, nastava matematike, kompetencije nastavnika

Specific Aspects of Applying a Model Based on a Tablet-Human Hybrid Model of Avatar in Face-to-Face University Mathematics Classes

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Abstract. Pairwork is a social form of teaching. Communication between paired students becomes a valuable exchange of information and active and effective work that contributes to better understanding and appropriation of course content. Cooperative learning improves mathematical achievement and attitudes towards mathematics.

The Tablet-Human Hybrid Model of Avatar (T-HHMA) has been tested in primary school. In the form of an avatar, the model enables classroom attendance and participation for absent students who are physically unable to attend due to illness, isolation, or other reasons. The model utilised a present classmate as an agent who paired his tablet with the tablet of an absent student via an audio-video connection. The agent's role was to fulfil the absent student's requests. The application of this model has never been tested in university classrooms.

The new model, which is still in the development phase, introduces pairwork by pairing the absent student with his or her agent, thus opening up the possibility of taking advantage of cooperative learning.

The application of this model in solving mathematical problems is considered. Data are collected through participant observation and interviews.

Keywords: avatar, cooperative learning, model, pairwork, university face-to-face classes

1. Introduction

The recent announcement of the end of the Covid-19 pandemic does not diminish the need to enhance new hybrid teaching models. If anything, the sudden and unexpected crisis has taught us the importance of proactively preparing for rapid changes in the education system. Involuntary absences from the classroom will continue to occur, causing detrimental effects on absent students, their classmates, and the teaching process.

Developed initially as a timely response to the unfolding situation, the Tablet-Human Hybrid Model of Avatar (T-HHMA) exhibited adaptability that paved the path for new research endeavours. Stemming from the challenges posed by the pandemic, it now presents itself as an effective solution to addressing the absence of students from classes, whether due to illness or other circumstances.

There is a research gap between studies on models involving teleconferencing robots that are immobile or difficult to move and research on hybrid teaching models that require special teacher preparation or on-the-spot improvisation, highlighting the need for further investigation in this area.

This paper is a part of a more extensive research which aims to mitigate adverse effects on absent students and focuses on exploring further possibilities for improving the T-HHMA model and building a new and enhanced version.

2. Theoretical background

Aucejo and Romano (2014) indicate that extending the school calendar by ten days would increase maths and reading test scores by only 0.8 % and 0.2 % of a standard deviation, respectively, while a similar reduction in absences would lead to increases of 5.8 % and 3 % in maths and reading. In later research, authors concluded that extending the school calendar by ten days would increase maths and reading test scores by only 1.7 % and 0.8 % of a standard deviation, while a similar reduction in absences would lead to increases of 5.5 % and 2.9 % in maths and reading (Aucejo & Romano, 2016). Higher class attendance positively correlates with academic success and improved learning outcomes (Kassarnig et al., 2017; Stanca, 2006; Tetteh, 2018). There are also other problems related to absence from classrooms, such as difficulty making friends, loss of self-confidence and self-esteem of absent students, distancing from friends, the feeling of being let down by the project partners, disruption of classes upon return, etc. (Malcolm et al., 2003).

Pairwork is a social form of teaching. Students' informal chatter transforms into productive conversations and active participation, significantly affecting class discipline (Jurčić, 2012).

Benefits were explored in various fields, such as pair programming (Hanks et al., 2011) and greater grammatical and lexical accuracy (Dobao, 2012).

The implementation of cooperative learning approaches has been shown to lead to improved attitudes towards mathematics and higher academic achievement compared to traditional teaching methods (Abd Algani & Abu Alhaija, 2021; Ifamuyiwa & Akinsola, 2008; Tarim & Akdeniz, 2007; Zakaria et al., 2010).

During the Covid-19 pandemic, the Tablet-Human Hybrid Model of Avatar (T-HHMA) was developed and tested in a primary school setting. This model provides a solution for students who cannot attend class physically, allowing them to participate through an avatar actively. By pairing their tablets via audio-video communication, a designated classmate acts as an agent, fulfilling the absent student's requests and ensuring their engagement in the classroom (Duka et al., 2022).

3. Research methodology

The need to mitigate the negative impact on absent students determines the research goal: Collect qualitative data on the updated T-HHMA used in university face-to-face classes as a means for its further development.

The primary methods used to achieve the research goal were the modelling method and the interview.

Other scientific methods were used, such as qualitative analysis, questionnaire, systematic observation method in the natural environment, abstraction, generalisation, etc.

The study was conducted at the Faculty of Education in Osijek in courses presented in Table 1.

The teaching sessions, two consecutive in each course group, comprised lectures, exercises, and student presentations and will be called Sessions 1 to 8 in this paper.

The research questions are categorised into four groups to accomplish the desired objective, reflecting the participants' viewpoints and opinions regarding the subject matter:

- Inhabitors (absent students) – models functionality, quality of communication, quality of participation in classes, problems and success in solving mathematical tasks
- Agents – models functionality, quality of communication, workload, problems and success in solving mathematical tasks
- Classmates – disturbing and distracting
- Teachers – workload, disturbing and distracting

Additional remarks were opened for all groups. Responses were collected through interviews, group interviews, and questionnaires from Inhabitors and Agents.

The researcher's proximity to the classrooms during the study allowed for the timely resolution of technical issues, resulting in improved efficiency in several cases.

Teachers were instructed to prepare for classes in the usual way. All teachers, agents and inhabitors in this study were volunteers, and the model was presented and explained to them.

4. The course of the research

The internet connection quality was tested before the lectures using the online tool *Speedtest by Ookla* (`https://www.speedtest.net`). The communication software used in Session 1 was *BigBlueButton* and Zoom Cloud Meeting. Due to various problems, in the following lectures, it was replaced with Wire Version 3.30.4368, which is specialised in retaining connection in poor quality internet conditions.

To improve auditory quality for the inhabitors, researchers introduced mandatory usage of headsets. Bluetooth headsets were assigned to the agents starting from Session 3.

Agent A2 actively participated but was not interviewed. No students volunteered for the role of inhabitor I13 and agent A13.

To minimise disruption to the teaching process, discreet observations were conducted by the researcher at intervals of 15–20 minutes.

Table 1 presents the research progression, highlighting each session's unique aspects.

Table 1. The research timeline.

<table>
<tr><th>Course</th><th colspan="4">Teaching Mathematics II</th><th colspan="4">Mathematics in Play and Leisure</th><th colspan="7">Mathematical Culture and Communication</th></tr>
<tr><td>Year</td><td colspan="4">4th</td><td colspan="4">1st</td><td colspan="7">2nd</td></tr>
<tr><td>Study programme</td><td colspan="4">Class teacher study</td><td colspan="11">Early and pre-school education</td></tr>
<tr><td>Session</td><td colspan="2">1</td><td colspan="2">2</td><td colspan="2">3</td><td colspan="2">4</td><td colspan="2">5</td><td colspan="2">6</td><td>7</td><td colspan="2">8</td></tr>
<tr><td>Communication software</td><td colspan="2">BBB
Zoom</td><td colspan="13">Wire</td></tr>
<tr><td>Inhabitors using headsets</td><td colspan="4">No</td><td colspan="11">Yes</td></tr>
<tr><td>Internet connection</td><td colspan="4">Poor
(0.5-2 Mbit/sec)</td><td colspan="4">Excellent
(>80 Mbit/sec)</td><td colspan="4">Extremely poor
(<0.5 Mbit/sec)</td><td colspan="3">Excellent
(>80 Mbit/sec)</td></tr>
<tr><td>Com. device positioning</td><td colspan="4">Front rows</td><td colspan="4">Middle rows</td><td colspan="7">Front rows</td></tr>
<tr><td>Inhabitors</td><td>I1</td><td>I2</td><td>I3</td><td>I4</td><td>I5</td><td>I6</td><td>I7</td><td>I8</td><td>I9</td><td>I10</td><td>I11</td><td>I12</td><td>I14</td><td>I15</td><td>I16</td></tr>
<tr><td>Agents</td><td>A1</td><td></td><td>A3</td><td>A4</td><td>A5</td><td>A6</td><td>A7</td><td>A8</td><td>A9</td><td>A10</td><td>A11</td><td>A12</td><td>A14</td><td>A15</td><td>A16</td></tr>
</table>

The new model based on the Tablet-Human Hybrid Model of Avatar used in this study, illustrated in Figure 1, is still in development.

This model introduces pairwork by pairing the absent student with his or her agent. Also, it uses a broader set of devices with these attributes:

— Lightweight and portable
— Flexible (rotation, positioning, multiple cameras)
— Different devices on the inhabitor and agent sides

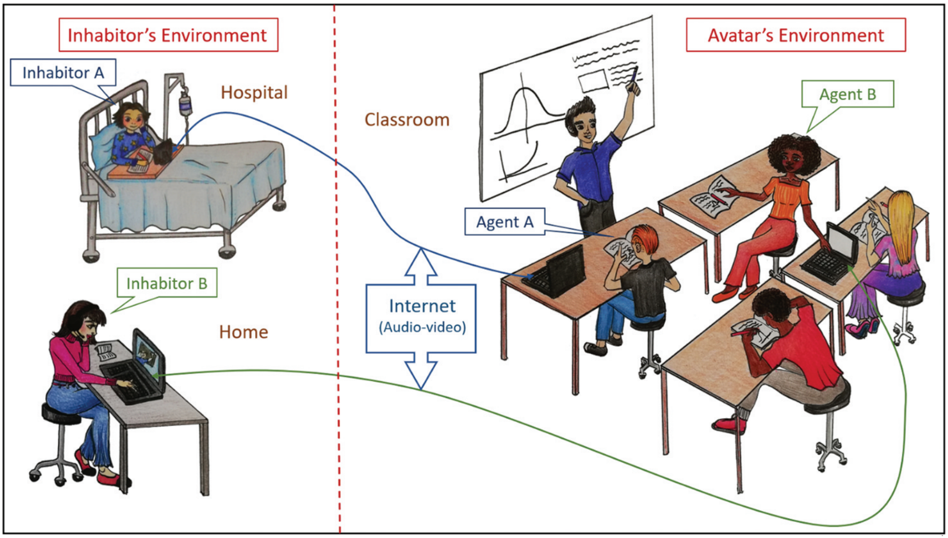

Figure 1. The new model in a real-life classroom. Illustrations courtesy of Marina Duka.

5. Discussion and results

All inhabitors and agents expressed their satisfaction with the new model, highlighting its functionality and frequently using terms like 'beneficial' and 'practical' to describe their experience, except for agent A4. Moreover, they have appreciated the importance of this model as it enables students who are physically prevented from attending to still be part of the classroom experience. Teachers embraced the presence of inhabitors and actively engaged in direct communication with them on numerous occasions.

Teachers reported no increase in workload. Classmates and teachers did not report any distractions throughout the sessions, except for minor disruptions during Session 1 caused by the ringtone during reconnection. In Session 1, there were five or more disconnections. However, in Sessions 5 and 6, where the internet quality was inferior, only one disconnection was reported, likely due to the attributes of the communication software.

The positioning of the communication device became more apparent in larger university classrooms compared to primary school classrooms. During Sessions 3 and 4, both inhabitors and agents faced frequent challenges with background noise caused by the placement of the communication device among their classmates. Inhabitors mentioned that the clarity of the teacher's voice varied based on the distance and volume. Through careful observation, the researcher noted a clear relationship between the number of students in the session and the background noise level.

In Session 2, the researcher witnessed a proactive approach of A3 in addressing the visibility issue experienced by her inhabitor I3. As the task involved intricate graphics and the internet connection was subpar, I3 encountered difficulties perceiving the task. However, A3 demonstrated resourcefulness by hand-drawing a graphic on paper and placing it near the camera. This innovative solution ensured optimal visibility for I3 without causing any disruptions to the teacher or classmates.

Most inhabitors and nearly half of the agents see the quality of internet connection as the most influential on the quality of classes because of the high impact on video quality. Switching to a different communication software resulted in minimal impact on audio quality.

Video resolution was in the range of 1080p down to 144p (reported by inhabitor I10), which is 56.25 times worse. I12 expressed herself most vividly, describing her seeing a “white blur instead of a whiteboard”.

6. Specifics of solving mathematical tasks in new learning environment

Inhabitor I5 and agents A5 and A6 noticed a problem with the lack of teaching aids and materials for inhabitors. The model's capabilities do not extend to unplanned absences; it ultimately relies on the student's ingenuity and effort. In the case of the planned absence the solution is the digital preparation of teaching aids and materials, but that would cause a workload increase for teachers.

Agents reported an overall success rate of 87 % in solving mathematical tasks. Out of the 12 agents who responded, 10 successfully completed 80 % or more of the tasks, with none reporting a completion rate of 20 % or less. Inhabitors reported an overall success rate of 69 % in solving mathematical tasks. Among the 15 inhabitors, 9 completed 80 % or more of the tasks, while 3 reported a completion rate of 20 % or less. These lower completion rates were observed in sessions 5 and 6, with deplorable internet conditions. The diminished task-solving achievement among the inhabitors (mean $= 17.5$) in contrast to their agents (mean $= 90$) under those internet conditions is further corroborated by the t-test results ($t = 5.256$, $p = 0.002$). No statistically significant differences were observed between inhabitors and agents in the other sessions.

Visual representations are essential in the learning of mathematics. According to Lowrie et al. (2012), variations in the graphic element had an influential and generally positive effect on student performance and sense-making. When the graphic element of tasks was modified, many students who had incorrectly solved tasks could reason more sophisticatedly about the nature and content of the tasks.

By contrast, changes in the task to the text or symbols had only a small effect on student performance and sense-making.

> "...New curricular emphases and approaches, innovative classroom practices and the understandings we develop from them, re-value visualisation and its nature, placing it as a central issue in mathematics education. ... understanding it better should certainly enrich our grasping of aspects of people's sense-making of mathematics, and thus serve the advancement of our field" (Arcavi, 2003, p. 238).

Responders reported that visual issues were commonly experienced during sessions 3 to 6. The cause of these issues in sessions 3 and 4 was related to the placement of the communication device in the middle rows, resulting in a considerable distance between the device and the teacher or school board. However, in sessions 5 and 6, the visual problems were primarily due to the inferior internet quality, and the situation the inhabitors faced was similar to the experience of visually impaired individuals in a classroom.

Spinczyk et al. (2018), supported by Brzoza et al. (2012) and Brzoza & Maćkowski (2014), highlighted an alternative method for learning mathematics among the visually impaired that was proposed at the Silesian University of Technology, Poland. The following elements characterise the proposed method:

1. Dividing the exercise into elementary stages.
2. Checking the results at individual stages.
3. Entering user interactions.
4. Providing contextual support through the selection of theories and hints.
5. Providing alternative presentations with descriptions containing structural elements.

Adapting the method and its elements and implementing them in agent-inhabitor communication under satisfying internet conditions has the potential to yield better results. A short set of instructions could be primed as a leaflet or tutorial on the communication device, but a more profound study is required for a solid conclusion.

Although the new model is conceived to be applicable to unplanned rather than planned situations, the authors intend to conduct research to identify specific differences in solving various types of mathematical problems during pre-prepared lessons, taking into account this new learning environment.

7. Conclusion

Success in solving mathematical tasks using the new model depended on several factors: quality of internet connection, positioning of the communication device in the classroom, group and classroom size, communication software properties, and student ingenuity and effort.

Mandatory headsets for inhabitors should be implemented in the next iteration of the model.

Communication devices should be positioned in the front rows whenever it is possible for better results.

Communication software's ability to maintain a stable connection becomes crucial, especially in instances of poor internet quality.

Further research should be conducted to explore novel methods or solutions that can be effectively utilised in challenging internet conditions.

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Specifičnosti primjene modela zasnovanog na hibridnom humano-tabletnom modelu avatara u fakultetskoj nastavi matematike uživo

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Sažetak. Rad u paru jedan je od socijalnih oblika nastave. Komunikacija između uparenih studenata postaje važna razmjena informacija te aktivan i učinkovit oblik rada koji pridonosi boljem razumijevanju i usvajanju sadržaja kolegija. Suradničko učenje poboljšava matematička postignuća i stavove prema matematici.

Hibridni humano-tabletni model avatara (T-HHMA) testiran je u osnovnoj školi. U obliku avatara, model omogućuje pohađanje nastave i sudjelovanje odsutnih učenika koji nastavi zbog bolesti, izolacije ili drugih razloga nisu u mogućnosti prisustvovati. U modelu se kao agent koristi učenik iz razreda koji je audio-video vezom upario svoj tablet s tabletom odsutnog učenika. Uloga agenta bila je ispunjavanje zahtjeva odsutnog učenika. Primjena ovog modela nije do sada testirana u sveučilišnoj nastavi.

Novi model, koji je još u fazi razvoja, uparujući odsutnog studenta sa svojim agentom uvodi socijalni oblik rada *rad u paru*, čime se stvara mogućnost iskorištavanja prednosti suradničkog učenja.

Razmatrana je primjena ovog modela pri rješavanju matematičkih zadataka. Podatci su prikupljeni metodom promatranja sudionika i intervjuiranjima.

Ključne riječi: avatar, model, rad u paru, sveučilišna nastava uživo, suradničko učenje

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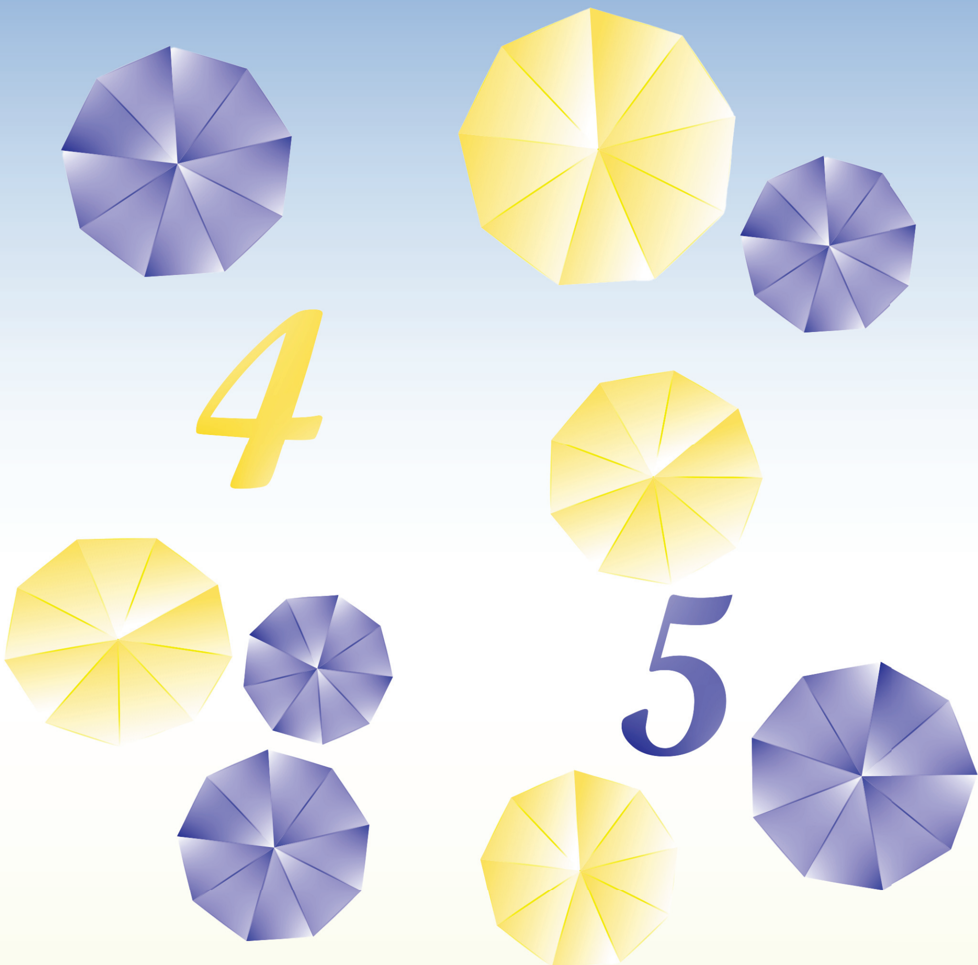

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